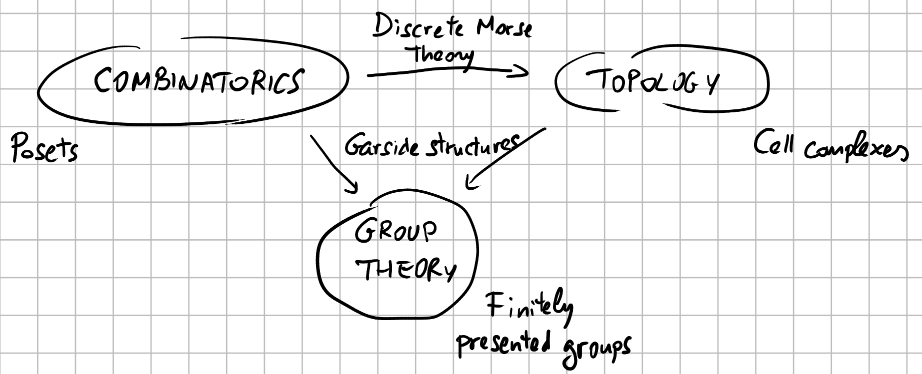
Combinatorial Topology and Group Theory

Lecture I
22/11/2021



Posets

[Stanley, Chapter 3]

Let $(P, \leq)$ be a partially ordered set (poset).

<u>Def</u> A pair (x,y) is a <u>cover relation</u> if $x<y$ and there is no z such that $x<z<y$.

We write $x \lessdot y$.

<u>Def</u> A <u>chain of length m</u> is a totally ordered subset $\{x_0 < x_1 < \dots < x_m\}$ of $m+1$ elements.

<u>Def</u> The <u>length</u> of P is the supremum of the lengths of all chains.

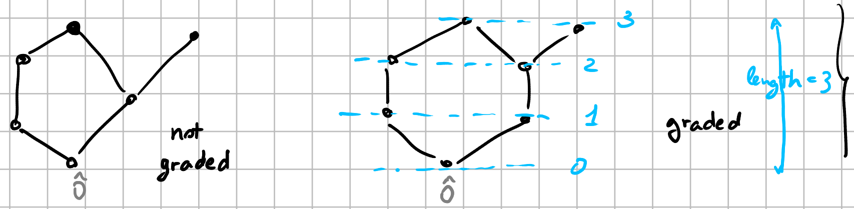
<u>Assumption</u> The cover relations determine $\leq$. $\quad$ (Counterexample: $(\mathbb{R}, \leq)$)

$\Uparrow$ exercise

Sufficient condition: All intervals $[x,y] = \{z \in P \mid x \leq z \leq y\}$ have finite length.

<u>Hasse diagram of P</u>

Each edge $\overset{y}{\underset{x}{|}}$ is a cover relation $x \lessdot y$.



both are meet-semilattices but not join- ...

<u>Def</u> P is <u>graded</u> (ranked) if there is a rank function $\rho: P \to \mathbb{Z}$ such that $\rho(y) = \rho(x) + 1$ $\forall\, x \lessdot y$

$\Uparrow$ exercise

Sufficient condition: all <u>maximal</u> chains have the same (finite) length.

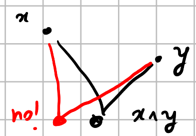
maximal: w.r.t. inclusion

<u>Def</u> The <u>bottom</u> / <u>top</u> element of P is the unique minimal / maximal element (if it exists)

Notation: $\hat{0}$, $\hat{1}$

P is <u>bounded</u> if it has both bottom and top.

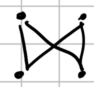
Def P is a <u>meet-semilattice</u> if every pair $x, y \in P$ has a unique maximal lower bound $x \wedge y$ (<u>meet</u>)



<u>join-semilattice</u> minimal upper bound $x \vee y$ (<u>join</u>)

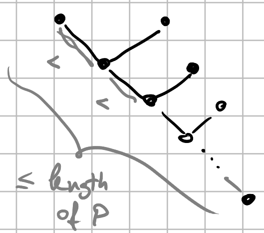
If both, P is a <u>lattice</u>

Non-example:



Remark If P has finite length

- meet-semilattice $\Rightarrow$ $\exists$ bottom
- join-semilattice $\Rightarrow$ $\exists$ top
- lattice $\Rightarrow$ P is bounded.

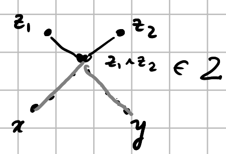


Theorem If P is a finite-length meet (join)-semilattice and has a top (bottom) element, then it is a lattice.

Proof Let $x, y \in P$. Let $Z = \left\{ z \in P \;\middle|\; \begin{matrix} z \geq x \\ z \geq y \end{matrix} \right\} \ni \hat{1}$. Then Z is closed w.r.t. $\wedge$

$\Rightarrow$ Z is a meet-semilattice

$\Rightarrow$ Z has a bottom element, $x \vee y$.

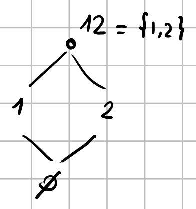


$z_1, z_2 \in Z$

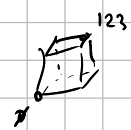
$\square$

Example 1 Boolean lattice $B_n = \left(2^{\{1,\dots,n\}}, \subseteq\right)$.

$n = 2$



In general, the Hasse diagram is the 1-skeleton of an n-cube

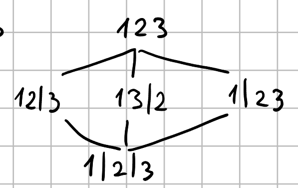


It is a product of n copies of 

- Meets and joins are given by $\cap$ and $\cup$.
- Graded by $\rho(X) = |X|$.

Example 2 Partition lattice Π_n, the poset of all set partitions of $\{1, \dots, n\}$ ordered by refinement

$n = 3$

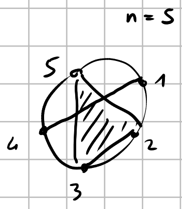


- $\pi = \{\underbrace{B_1, \dots, B_k}_{\text{blocks}}\}$ where $B_1 \sqcup \dots \sqcup B_k = \{1, \dots, n\}$ and $B_i \neq \emptyset$
- $\pi_1 \leq \pi_2$ if all blocks of π_1 are contained in some block of π_2.

- Meet $\pi_1 \wedge \pi_2$: i, j are in the same block iff they are in the same block in π_1 and π_2
- Join $\pi_1 \vee \pi_2$: ...
- Graded: $\rho(\pi) = n - |\pi|$.

Example 3 Noncrossing partition lattice $NC_n \subseteq \Pi_n$

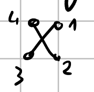
$\pi \in \Pi_n$ is noncrossing if the convex hulls of the blocks are disjoint.



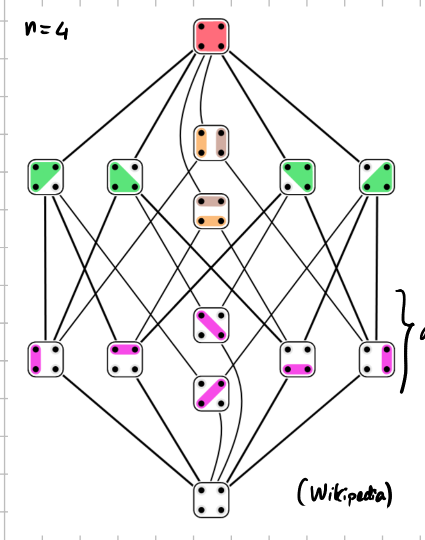
14 | 235

crossing!

Missing:



13|24



(Wikipedia)

- Graded
- Meets are the same as in Π_n
  + top element $\Rightarrow$ NC_n is a lattice
- Joins are not the same:
  $$13|2|4 \vee 24|1|3 = \begin{cases} 13|24 & \text{in } \Pi_4 \\ 1234 & \text{in } NC_4 \end{cases}$$

Exercise:
- $|NC_n| = Cat(n) = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$
- $|\{\pi \in NC_n \mid \rho(\pi) = k\}| = \underbrace{Nar(n,k)}_{\text{Narayama}} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

More: [Armstrong]

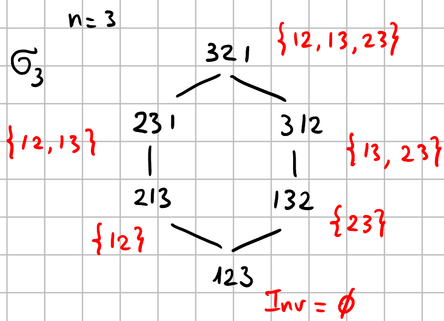
Example 4 Weak order on $\mathfrak{S}_n$ (symmetric group)

Write permutations $u \in \mathfrak{S}_n$ as n-words $u(1)\,u(2)\dots u(n)$

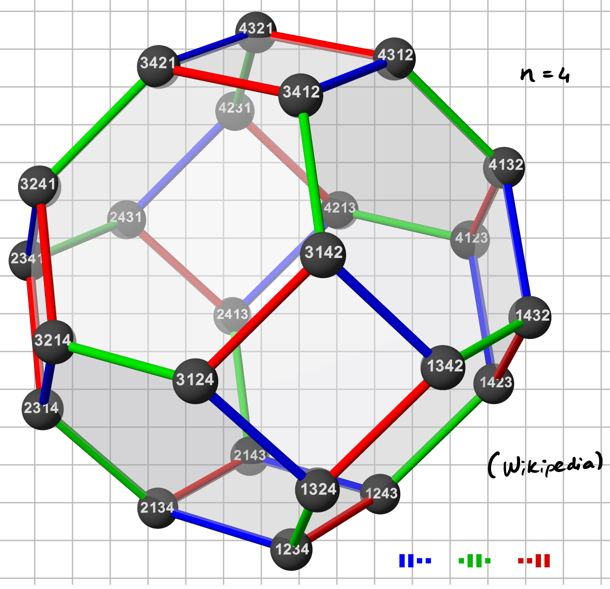
$u \leq v$ if every "inversion" in u is also an inversion in v

$1 \leq i < j \leq n$ such that j comes before i

$Inv(u) \subseteq Inv(v)$



- Cover relations: swap two consecutive numbers.
- Graded by $\rho(u) = |Inv(u)|$
  $\Rightarrow$ length of $\mathfrak{S}_n$ is $\binom{n}{2}$
- Hasse diagram of $\mathfrak{S}_n$ is the 1-skeleton of a permutahedron



(Wikipedia)

- Lattice! Idea:
  - X is an inversion set iff X and X^c are "transitive"
  - Define $u \vee v$ by $Inv(u \vee v) := \overline{Inv(u) \cup Inv(v)}$ (exercise)

- NC_n and $\mathfrak{S}_n$

	NC_n	$\mathfrak{S}_n$
# atoms	$\binom{n}{2}$	$n-1$
length	$n-1$	$\binom{n}{2}$

W finite Coxeter group
- $R =$ {simple reflections}
- $R =$ {all reflections}, $[1, \text{Coxeter element}]$

Example 5 Lattice of linear subspaces in a finite-dim. vector space V over $\mathbb{F}$, ordered by (reversed) inclusion.

$$\mathrm{Sub}(V)$$

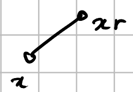
- Graded by (co)dimension
- Meets and joins are $\cap$ and $+$
- B_n and Π_n are naturally $\subseteq \mathrm{Sub}(\mathbb{F}^n)$

$$\{i_1, \dots, i_k\} \mapsto \{x_{i_1} = x_{i_2} = \dots = x_{i_k} = 0\} \in \mathrm{Sub}(\mathbb{F}^n)$$

$$\pi = \{\beta_1, \dots, \beta_k\} \mapsto \{x_i = x_j \text{ whenever } i, j \text{ are in the same block}\}$$

Example 6 G group, $R = R^{-1}$ generating set $(1 \notin R)$

$x \le y$ if there is a geodesic from 1 to y passing through x in the (right) Cayley graph of G w.r.t. R.



$r \in R$

- $\hat{0} = 1$
- Graded by length function
- $\hat{1}$ doesn't have to exist ($\mathbb{Z}_3 =$ )

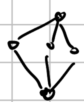
However, we can consider an interval $[1, g]$ for some $g \in G$.

Question: Which examples 1–5 are a particular case of 6?

① Boolean B_n ✓
② Partition Π_n ×
③ Noncrossing P. NC_n ✓
④ $\mathfrak{S}_n$ ✓
⑤ $\mathrm{Sub}(V)$ ✓ in some cases

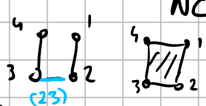
① $G = \mathbb{Z}_2^n$, $R = \{(1,0,\dots,0), \dots\}$

④ $G = \mathfrak{S}_n$, $R = \{(12), (23), \dots, (n-1\ n)\}$

(⑤ $V = \mathbb{F}_2^2$ $\mathrm{Sub}(V) =$  cannot be a whole group)

③ $G = \mathfrak{S}_n$, $R = \{\text{all transpositions}\}$

$NC_n = [1, (12\dots n)]$



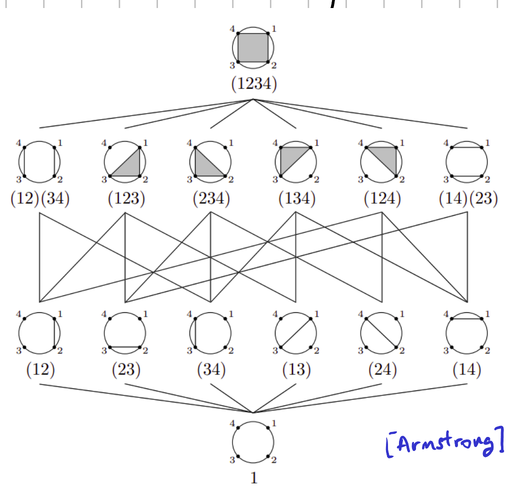
Example: $\mathbb{R}^n$

⑤ $\mathrm{char}\ \mathbb{F} \neq 2$ and V has an anisotropic quadratic form Q

$G = O(V, Q)$, $R = \{\text{all reflections}\}$

$[\mathrm{id}, -\mathrm{id}] \cong \mathrm{Sub}(V)$ (Exercise)

$f \longmapsto \mathrm{Ker}(f - \mathrm{id})$



[Armstrong]