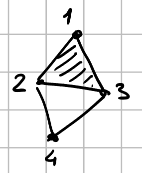
COMBINATORIAL TOPOLOGY AND GROUP THEORY

Lecture II
25/11/2021
(Happy Thanksgiving!)

CELL COMPLEXES [Hatcher]

Simplicial complexes

$X =$ 

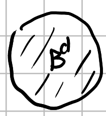
$\mathcal{F}(X) = \{ 123, 24, 34, \dots \}$ (not $\emptyset$)

maximal simplices

CW complexes

$X^{(0)} = \; \cdot \; \cdot \; \cdot \; \cdot$

$X^{(d)} =$ attach d-cells to $X^{(d-1)}$



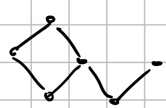
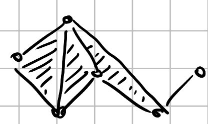
$\varphi_\sigma : \partial B^d \longrightarrow X^{(d-1)}$ attaching map of σ

$\partial B^d = S^{d-1}$

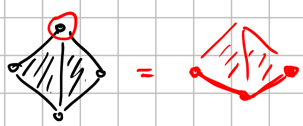
$\Phi_\sigma : B^d \longrightarrow X^{(d)}$ characteristic map of σ

Posets $\underset{\text{face poset}}{\overset{\text{order complex}}{\rightleftarrows}}$ Cell complexes

Def The order complex of a poset P is the simplicial complex with vertex set P and one simplex for every non-empty chain

- $P =$  $\Delta(P) =$ 

- $P = B_n$ Boolean lattice $\Delta(P) =$ triangulation of the n-cube with $n!$ n-simplices

$n = 2$ 

$n!$ maximal chains

Remark If P has $\hat{0}$ or $\hat{1}$ then $\Delta(P)$ is a cone (often one looks at $P \setminus \{\hat{0}, \hat{1}\}$).

Def The face poset $\mathcal{F}(X)$ of a CW complex X is the set of cells with $\tau \leq \sigma$ if $\tau \subseteq \overline{\sigma}$

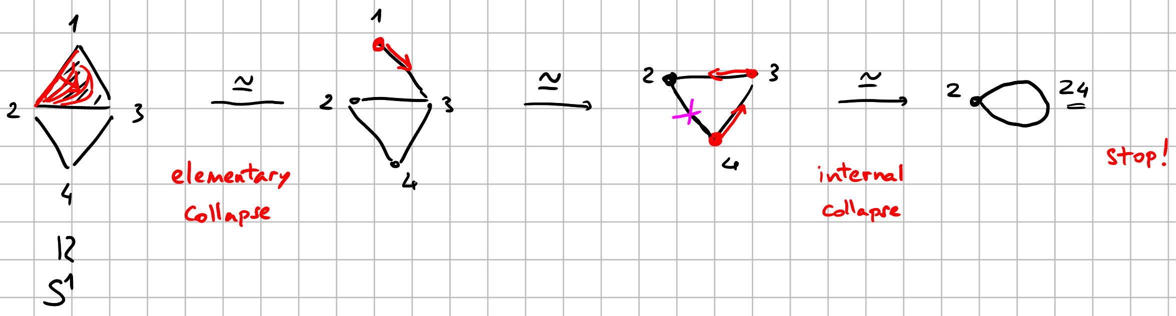
Technicality: $X^{(1)} =$ 

$\overline{\mathcal{F}}(X) =$ the set of cells ordered by $\tau \leq \sigma$ if $\tau \cap \overline{\sigma} \neq \emptyset$ (+ transitive closure)

DISCRETE MORSE THEORY

[Batzies, Chapter 3 ; Forman]

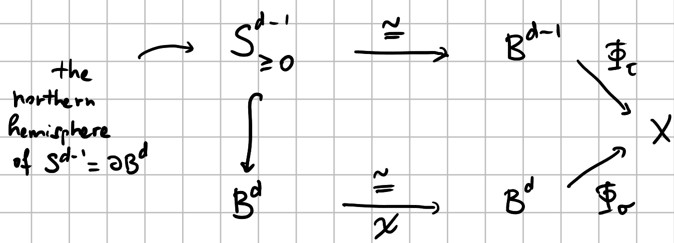
<u>Example</u>



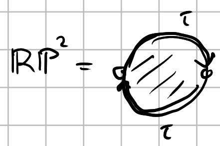
<u>Def</u> Let X be a CW complex, σ a d-cell, $\tau < \sigma$ a $(d-1)$-cell (in $\partial(\sigma)$)

τ is a <u>regular face</u> of σ if:

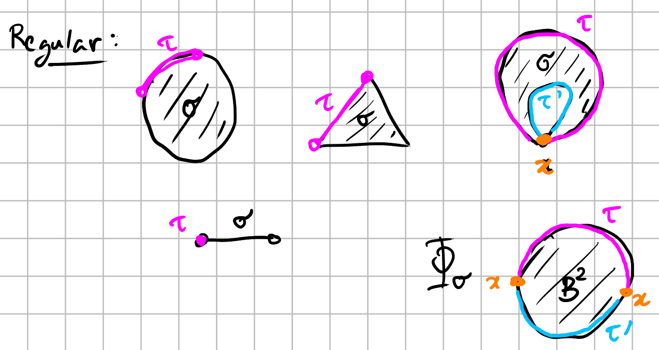
- the following diagram commutes: (for some homeomorphisms $\cong$)



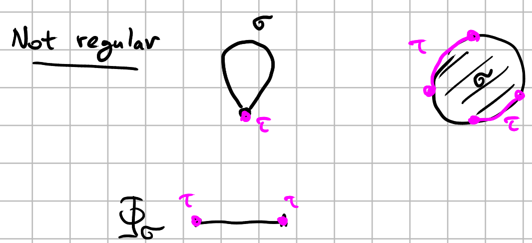
- $(\Phi_\sigma \circ \chi)^{-1}(\tau) = S^{d-1}_{>0}$.



<u>Regular:</u>



<u>Not regular</u>

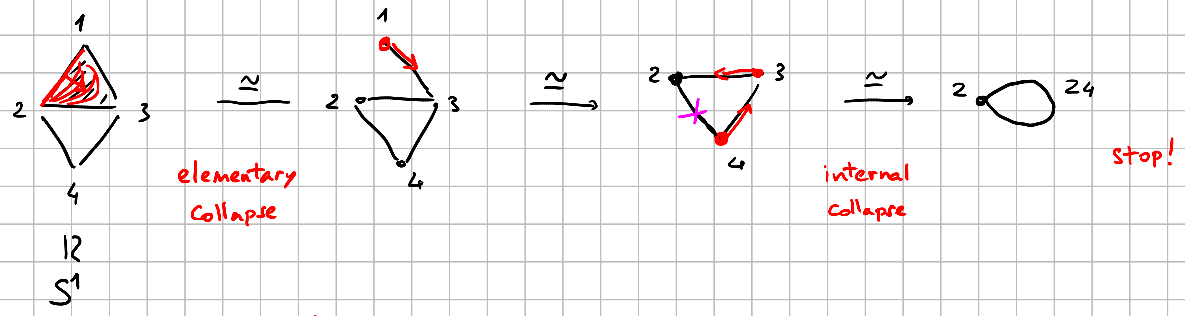


<u>Elementary collapse</u> τ regular face of σ and $\sigma \cup \tau$ does not intersect any closed cell other than $\bar{\sigma}$

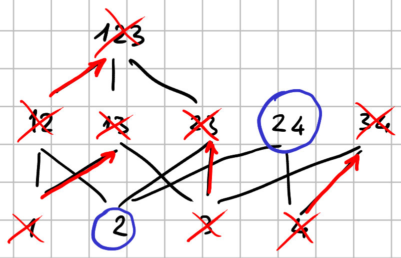
then there is a deformation retraction

$$h_{\tau \to \sigma} : X \longrightarrow X \setminus (\sigma \cup \tau)$$

Example

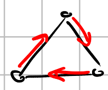


$\mathcal{F}(X) =$



The red arrows define a matching on (the Hasse diagram of) $\mathcal{F}(X)$.

We need to avoid



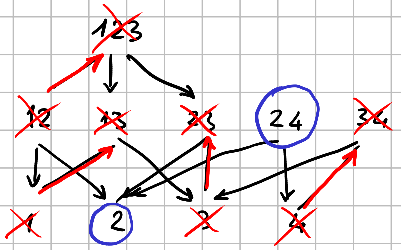
! $\neq \emptyset$

Def Let $\mathcal{M}$ be a subset of the set of edges of the Hasse diagram of $\overline{\mathcal{F}}(X)$ such that τ is a regular face of σ $\forall (\tau < \sigma) \in \mathcal{M}$.

- $\mathcal{M}$ is a <u>matching</u> if its edges are pairwise disjoint.
- Let $G_X^{\mathcal{M}}$ be the Hasse diagram of $\overline{\mathcal{F}}(X)$ with oriented edges:

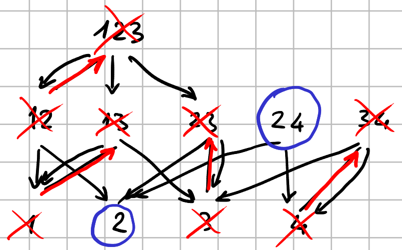
$$\sigma \downarrow \tau \quad \text{if } (\tau < \sigma) \notin \mathcal{M} \quad \text{and} \quad \sigma \uparrow \tau \quad \text{if } (\tau < \sigma) \in \mathcal{M}$$

$G_X^{\mathcal{M}} =$



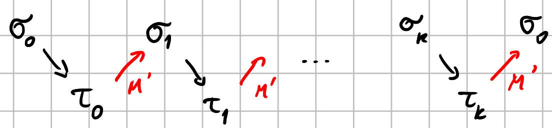
- $\mathcal{M}$ is acyclic if $G_X^{\mathcal{M}}$ is acyclic (no directed cycles).
- A cell σ is <u>critical</u> if it does not occur in any edge of $\mathcal{M}$
- Let $\tilde{G}_X^{\mathcal{M}}$ be the graph $G_X^{\mathcal{M}}$ with $\sigma \updownarrow \tau$ if $(\tau < \sigma) \in \mathcal{M}$.

$\tilde{G}_X^{\mathcal{M}} =$



Lemma If M is an acyclic matching and $M' \subseteq M$, then M' is also acyclic.

Proof Suppose there is a cycle in $G_X^{M'}$:



"alternating cycle"

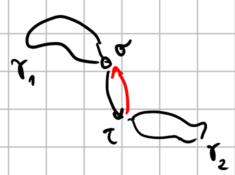
The dimension decreases along ↓ and increases by 1 along ↑
$\Rightarrow$ decreases by 1 along ↓.

This is also a cycle in G_X^M! Contradiction □

Lemma If M is an acyclic matching, then the only cycles in $\tilde{G}_X^M$ are $\sigma \uparrow\downarrow \tau$

Proof A cycle γ in $\tilde{G}_X^M$ must contain at least one $\sigma \uparrow\downarrow \tau$, otherwise γ would be a cycle in some $G_X^{M'}$ for some $M' \subseteq M$.

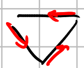
Then $\gamma =$



and γ_1, γ_2 are cycles $\Rightarrow$ by induction, γ is a union of $\sigma' \uparrow\downarrow \tau'$.

M is a matching, so you can have only one $\sigma \uparrow\downarrow \tau$. □

Example

- $X =$  is not acyclic  $= \mathcal{F}(X)$

- $\mathbb{R} =$  is acyclic! 

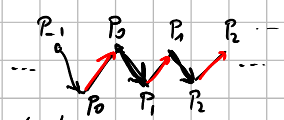
  $\mathbb{R} \not\simeq \emptyset$

  Problem: there is an infinite path in G_X^M!

Def Let $\theta : \mathcal{F}(X) \to P$ (any poset) be a poset map ("a grading of X by P")
$\tau \leq \sigma \Rightarrow \theta(\tau) \leq \theta(\sigma)$

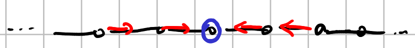
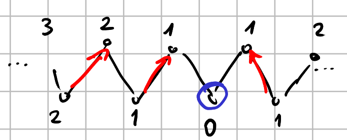
- θ is <u>compatible</u> with a matching M if $\theta(\sigma) = \theta(\tau)$ $\forall (\tau < \sigma) \in M$.
- θ is <u>proper</u> if $\theta^{-1}(P_{\leq p})$ is finite $\forall p \in P$
- M is <u>proper</u> if it is compatible with some proper grading θ.

Example

- $\mathbb{R} =$  is acyclic! 

  $\mathbb{R} \not\simeq \emptyset$

  $\cdots \geq p_0 \geq p_1 \geq p_2 \geq \cdots$

- $\mathbb{R} =$  

  $P = \mathbb{N}$ $\quad \theta^{-1}(\mathbb{N}_{\leq n})$ has $4n+1$ cells

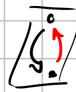
Main theorem of discrete Morse theory

Let X be a CW complex, M a proper acyclic matching on X.

① $X \simeq X_M$ (Morse complex) having cells in dimension-preserving bijection with the critical cells of X.

② If the critical cells form a subcomplex of $X_M \subseteq X$, then X deformation retracts onto X_M.

Def Given an acyclic matching M, its universal grading is the projection $\theta_M : \overline{\mathcal{F}}(X) \to P_M := \overline{\mathcal{F}}(X) / M$

where P_M is partially ordered because $\tilde{G}_X^M$ has  as the only cycles.

It's the finest grading compatible with M: for every $\theta : \overline{\mathcal{F}}(X) \to P$ compatible with M,

$$\overline{\mathcal{F}}(X) \xrightarrow{\theta_M} P_M \overset{\theta'}{\dashrightarrow} P, \qquad \overline{\mathcal{F}}(X) \xrightarrow{\theta} P$$

(θ' poset map).

Lemma If M is proper (compatible with some proper grading θ) then its universal grading is proper.

Proof We use the universal property.
Let $q \in P_M$. If $\theta_M(\sigma) \leq q$, then $\theta(\sigma) = \theta'(\theta_M(\sigma)) \leq \theta'(q)$ (θ' is order-preserving)

$\Rightarrow \sigma \in \theta^{-1}(P_{\leq \theta'(q)})$ which is finite because θ is proper. □

Construction of the Morse complex X_M

Given a proper acyclic matching M on X, we inductively construct CW complexes $(X_M)_{\leq p}$ and homotopy equivalences $h_{\leq p} : X_{\leq p} \to (X_M)_{\leq p}$. (For $p \in P_M$)

$X_{\leq p}$: the subcomplex with cells $\theta_M^{-1}((P_M)_{\leq p})$

$\theta_M : \overline{\mathcal{F}}(X) \to P_M$

Base step: if $p \in P_M$ is minimal, then it is a single critical cell (otherwise $p = (\tau < \sigma)$ regular face and σ has other faces)

$(X_M)_{\leq p} := X_{\leq p} = *$

$h_{\leq p} = id_*$