Combinatorial Topology and Group Theory

Lecture III
30/11/2021

<u>Induction step</u> Suppose we did everything for all $q < p$ (in P_M) and for all $q' \le q < p$ we have $(X_M)_{\le q'} \subseteq (X_M)_{\le q}$ and the following diagram commutes:

$$\begin{tikzcd} X_{\le q'} \arrow[r, "h_{\le q'}"] \arrow[d, hook] & (X_M)_{\le q'} \arrow[d, hook] \\ X_{\le q} \arrow[r, "h_{\le q}"] & (X_M)_{\le q} \end{tikzcd}$$

Let $(X_M)_{<p} := \bigcup_{q<p} (X_M)_{\le q}$ and let $h_{<p} : X_{<p} \to (X_M)_{<}$ be the union of all $h_{\le q}$ for $q<p$

Then $h_{<p}$ is a homotopy equivalence ([Brown, Theorem 7.4.3] + induction).

<u>Case I</u> : $p = (\tau <\cdot\, \sigma) \in M$.

Set $(X_M)_{\le p} := (X_M)_{<p}$ and extend $h_{<p}$ to $h_{\le p} = h_{<p} \circ h_{\tau \to \sigma}$ which is a homotopy equivalence.

<u>Case II</u> : p is a critical cell σ, with attaching map $\varphi_\sigma : \partial B^d \to X_{<p}$ $\left(X_{\le p} = B^d \cup_{\varphi_\sigma} X_{<p} \right)$

Set $(X_M)_{\le p} := B^d \cup_{h_{<p} \circ \varphi_\sigma} (X_M)_{<p}$ and $h_{\le p} = id_{B^d} \cup_{\varphi_d} h_{<p}$.

Then $h_{\le p}$ is a homotopy equivalence by [Brown, Theorem 7.5.7]

$$\begin{tikzcd} X_{<p} \arrow[d, "h_{<p}"'] & \partial B^d \arrow[l, "\varphi_\sigma"'] \arrow[r, hook] \arrow[d, "id"] & B^d \arrow[d, "id"] \\ (X_M)_{<p} & \partial B^d \arrow[l, "h_{<p} \circ \varphi_\sigma"'] \arrow[r, hook] & B^d \end{tikzcd}$$

homotopy equivalences

cofibrations

Now, define $X_M := \bigcup_{p \in P_M} (X_M)_{\le p}$ and $h = \bigcup_{p \in P_M} h_{\le p} : X \to X_M$.

<u>Lemma</u> $h : X \to X_M$ is a homotopy equivalence

<u>Proof</u> By Whitehead's theorem [Hatcher, Theorem 4.5], it is enough to prove that $h_* : \pi_n(X) \to \pi_n(X_M)$ is an isomorphism $\forall\, n \ge 1$.

<u>Injectivity</u> : Suppose $[f] \in \pi_n(X) \longmapsto 0$. $\quad f : (S^n, s_0) \to (X, x_0)$
Then $h \circ f \simeq *$ in X_M and the homotopy H takes values in a compact subspace of X_M

Similarly, $\mathrm{im}(f) \subseteq$ finite subcomplex of X

There exist $p_1, \dots, p_k \in P_M$ such that

$$\mathrm{im}(f) \subseteq X_{\le p_1} \cup \dots \cup X_{\le p_k} \quad \text{and} \quad \mathrm{im}(H) \subseteq (X_M)_{\le p_1} \cup \dots \cup (X_M)_{\le p_k}$$

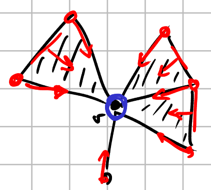
$$\xrightarrow[h_{\le p_1} \cup \dots \cup h_{\le p_k}]{\simeq}$$

$$\Rightarrow f \simeq *$$

Surjectivity Similar. $\square$

This proves the Main Theorem!

Example 1 X (simplicial complex) $=$ cone on a vertex v $\quad (\sigma \in \mathcal{F}(X) \Rightarrow \sigma \cup \{v\} \in \mathcal{F}(X))$



$M = \{ \tau <\cdot\ \tau \cup \{v\} \mid \tau \not\ni v \}$

Why acyclic?

An alternating cycle would be

$\tau_0 \nearrow \tau_0 \cup \{v\} \searrow \tau_1 \nearrow \tau_1 \cup \{v\}$

This edge does not exist! $\quad v \in \tau_1$

Example 2 $X^{(k)} =$ k-skeleton of a cone $\quad (k \ge 1)$

$M' = \{ \tau <\cdot\ \tau \cup \{v\} \mid \tau \not\ni v \text{ and } \dim(\tau) \le k-1 \}$

is acyclic in X because $M' \subseteq M$

Critical simplices: the vertex v and the k-dimensional simplices not containing v

$$\Rightarrow X_M = \bigvee_{\substack{\#\,k\text{-simplices} \\ \text{not containing } v}} S^k$$

For instance, if $X = \Delta^n$ (n-dim simplex) then $X^{(k)} \simeq \bigvee_{\binom{n}{k+1}} S^k$.

Patchwork Theorem Let $\theta : \mathcal{F}(X) \to P$ be a poset map.
If we have acyclic matchings M_p on all $\theta^{-1}(p)$ for $p \in P$ then

$$M = \bigcup_{p \in P} M_p \quad \text{is acyclic.}$$

Proof Let $\gamma = \sigma_0 \searrow \tau_0 \nearrow \sigma_1 \searrow \tau_1 \nearrow \dots \tau_k \nearrow \sigma_0$ be an alternating cycle. Then $\theta(\sigma_0) \ge \theta(\tau_0) = \theta(\sigma_1) \ge \theta(\tau_1) = \dots = \theta(\sigma_0)$

$\Rightarrow$ all are equalities $\Rightarrow$ the whole cycle lives in one of the fibers $\theta^{-1}(p)$ $\square$

Example 3: Evasiveness

Fix a simplicial complex $X \subseteq \Delta^{n-1}$ ← the full simplex on $\{1,\dots,n\}$

($\emptyset$ is a valid simplex)
$\emptyset \in \mathcal{F}(X)$

Someone chooses $\sigma \in \mathcal{F}(\Delta^{n-1})$. We can ask questions "is v in σ?"

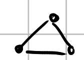
Task: understand if $\sigma \in \mathcal{F}(X)$.

Def X is *evasive* if any deterministic algorithm requires n questions for at least one σ.

Examples: $X = \Delta^{n-1}$ is non-evasive

$X = \cdot v$ " $\longrightarrow$ ask every vertex except v

$\mathcal{F}(X) = \{\emptyset, \{v\}\}$

$X =$  is evasive

Theorem X non-evasive $\Rightarrow$ X is collapsible (there is an acyclic matching with no critical simplices)

Proof Fix an algorithm A.

Given $\sigma \in \mathcal{F}(X)$ let $\theta(\sigma) \in \underbrace{\{YES, NO\}^{n-1}}$ be the sequence of answers

Totally ordered lexicographically with $NO < YES$

θ is a poset map!

$\tau < \sigma$

$\theta(\sigma) = (a_1, \dots, a_k, YES, \dots)$
$\theta(\tau) = (a_1, \dots, a_k, NO, \dots)$

↑ The first time you ask about $\sigma \setminus \tau$

Each fiber $\theta^{-1}(a_1, \dots, a_{n-1})$ is empty or $\{\tau <\cdot\, \sigma\}$ — differ by the last vertex we did not ask about $\square$

Consider the case $n = \binom{m}{2}$ $\rightsquigarrow$ Δ^{n-1} consists of all graphs on m labeled vertices.

Let $Z_m \subseteq \Delta^{n-1}$ be the simplicial complex of all disconnected graphs on m vertices

($m \geq 3$)

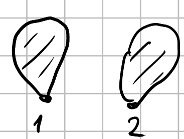
Proposition $Z_m \simeq \bigvee_{(m-1)!} S^{m-3}$ $\Rightarrow$ Z_m is evasive!

($\emptyset \notin Z_m$)

Proof (idea) Construct a matching as follows.

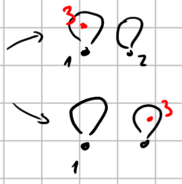
- Match all pairs $\tau <\cdot\, \tau \cup \{12\}$

The remaining simplices/graphs are:



2 components

- Match all pairs $\tau <\cdot \tau \cup \{13\}$, $\tau <\cdot \tau \cup \{23\}$

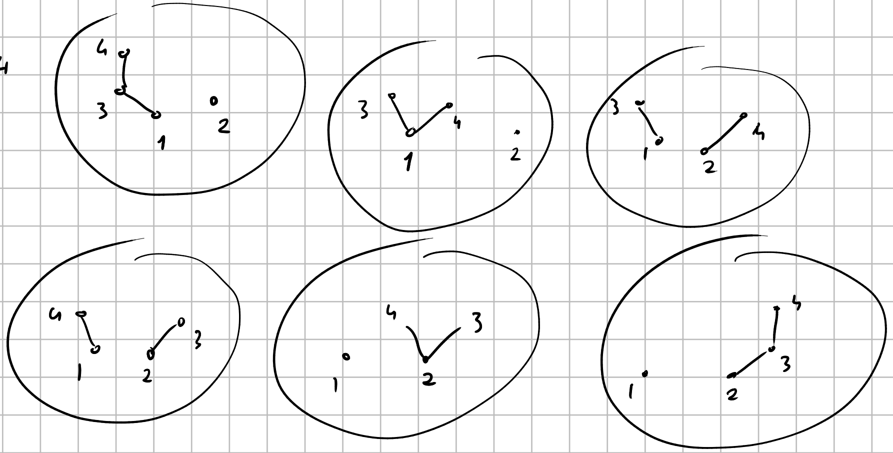


Remaining graphs:

...

At the end: disconnected trees with 2 components, 1 & 2 in different components and vertex numbers increase when you move away from 1 or 2

$m=4$



$\rightsquigarrow$ there are $(m-1)!$ such graphs, with $m-2$ edges ($(m-3)$-dim. simplices).

Exercise: acyclic!

Remark: $\Delta(\Pi_n \setminus \{\hat{0},\hat{1}\}) \simeq \bigvee_{(n-1)!} S^{n-3}$, not a coincidence!

[Kozlov, Chapters 11 and 13]

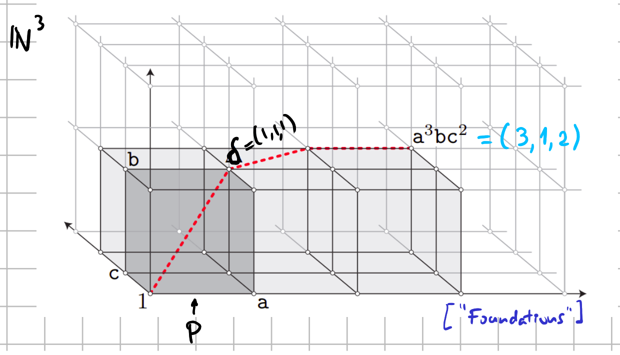
GARSIDE STRUCTURES

[McCammond, Dehornoy - Paris, "Foundations of Garside Theory", Charney - Meier - Whittlesey]

Example $\mathbb{Z}^n \supseteq \mathbb{N}^n \ni \delta = (1,\dots,1)$
$\cup\!|$
$P := \{0,1\}^n$ the poset of all divisors of δ in $\mathbb{N}^n$
(Boolean lattice) $x \le y$ if $xz = y$ for some $z \in \mathbb{N}^n$

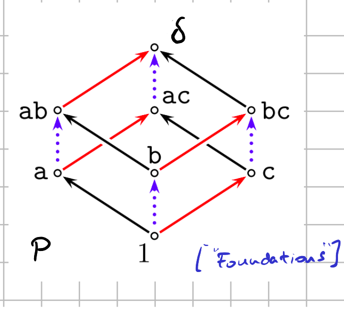
A normal form in $\mathbb{N}^n$: $x = x_1 x_2 \cdots x_k$, $x_i \in P$ is the $\gcd(\delta, x_i x_{i+1} \cdots x_k)$ (greatest lower bound)

$\mathbb{N}^3$



$a^3bc^2 = \underbrace{(abc)}_{\delta} \cdot (ac) \cdot a$

$a = (1,0,0)$
$b = (0,1,0)$
$c = (0,0,1)$



A normal form in $\mathbb{Z}^n$: $x = \delta^m \cdot x_1 x_2 \dots x_k$ $\quad m \in \mathbb{Z}$, $x_1 \dots x_n \in \mathbb{N}^n$ is in normal form and $x_1 \neq \delta$

Example. $x = (-3, 1, -2) = a^{-3} b c^{-2} = (abc)^{-3} \underbrace{b^4 c}_{\text{positive}} = \delta^{-3} (bc) \cdot b \cdot b \cdot b$

P labeled poset w/ axioms $\longrightarrow$ M monoid $\longrightarrow$ G group