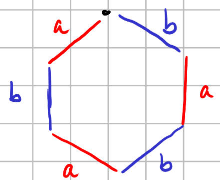
COMBINATORIAL TOPOLOGY AND GROUP THEORY

Lecture IV
03/12/2021

$(\mathfrak{S}_3, \text{ weak order})$



$a = (12)$
$b = (23)$

$NC_3 = [1, (123)]$ inside $\mathfrak{S}_3$
with generators

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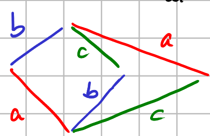
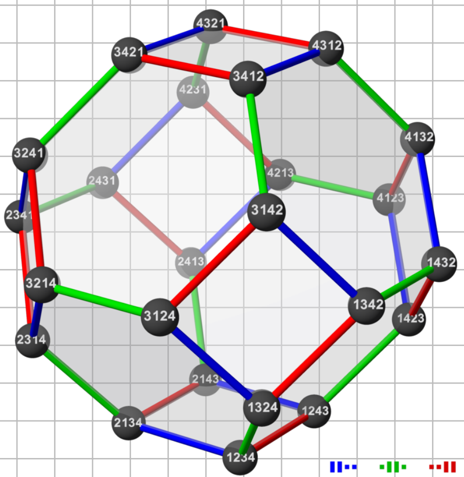
$c = (13)$

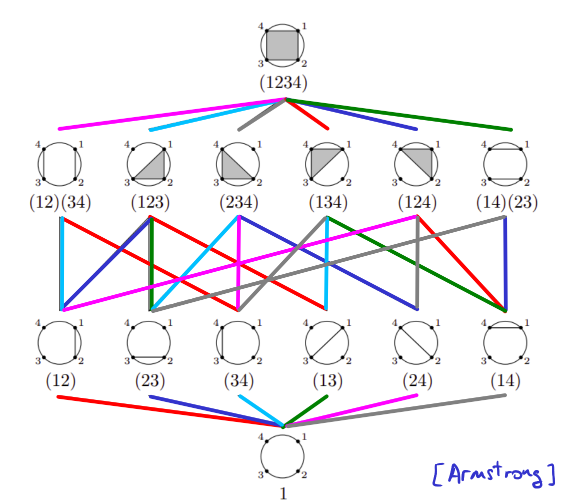


$(\mathfrak{S}_4, \text{ weak order})$

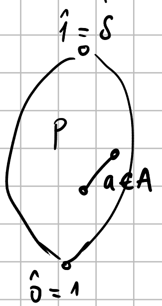


[Wikipedia]

$NC_4 = [1, (1234)]$ inside $\mathfrak{S}_4$

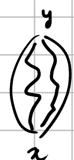


Let P be a finite bounded edge-labeled poset:



A = set of labels for edges of P (actually appearing at least once)

- $M = \langle A \mid \lambda(\gamma_1) = \lambda(\gamma_2)$ for any two maximal chains γ_1, γ_2 in some $[x,y] \rangle_+$ ← Monoid presentation

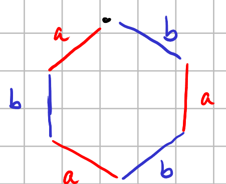
$[x,y] =$ 

$\gamma = \{ x_0 \overset{a_1}{<\cdot} x_1 \overset{a_2}{<\cdot} \dots \overset{a_k}{<\cdot} x_k \}$

$\lambda(\gamma) = a_1 a_2 \dots a_k \in A^*$

- G = group w/ same presentation

Example $(\mathfrak{S}_3, \text{ weak order})$



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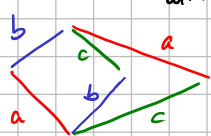
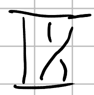


$G = \langle a, b \mid aba = bab \rangle$

braid group on 3 strands

$a =$  $b =$ 

$G = \langle a, b, c \mid ab = bc = ca \rangle$

$\downarrow$

the same group!

<u>Theorem</u> Let P be a <u>combinatorial Garside structure</u>. Then

- $P \hookrightarrow M \hookrightarrow G$
- There is a normal form $\delta^m x_1 \dots x_k$ with $x_i \in P$
- $K = \Delta(P)/_{\sim} \simeq K(G, 1)$
- Homological dim $(G) \leq \ell(P) \implies G$ torsion-free.

Given $x \leq y$ in P, let $\underline{\mathcal{L}(x,y)} = \{ \lambda(\gamma) \mid \gamma$ maximal chain in $[x,y] \}$

"language of $[x,y]$"

(C1) <u>Group-like</u>

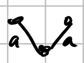
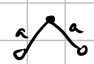
- Given $x \leq y \leq z$ and $x' \leq y' \leq z'$, if two pairs of languages are equal, then also the third pair of languages are equal

$\mathcal{L}(x,y) = \mathcal{L}(x',y')$ and $\mathcal{L}(y,z) = \mathcal{L}(y',z')$

$\implies \mathcal{L}(x,z) = \mathcal{L}(x',z')$

- $\forall x \in P$, $a \in A$, there is at most one edge $x <\cdot\, y$ labeled a
  " $z <\cdot\, x$ "

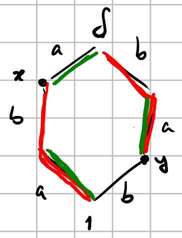
No:  

Sufficient condition: $P = [1, g]$ in the right Cayley graph of some group.

<u>Remark</u>/<u>Exercise</u>: (C1) $\implies$ if I know $\lambda(\gamma)$ for one chain of $[x,y]$, then $\mathcal{L}(x,y)$ is uniquely defined (induction on $\ell(\gamma)$)

(C2) Balanced $\{ \mathcal{L}(1,x) \mid x \in P \} = \{ \mathcal{L}(y,\delta) \mid y \in P \}$

Example



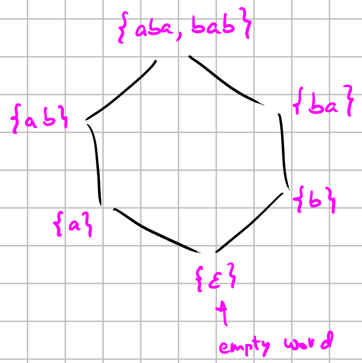
- $\mathcal{L}(1,x) = \{ ab \} = \mathcal{L}(y,\delta)$
- $\mathcal{L}(1,\delta) = \{aba, bab\}$

$= \{ \mathcal{L}(x,y) \mid x \leq y \text{ in } P \} =: L(P)$

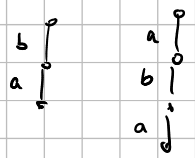
using (C1)

(Exercise)

$\Rightarrow$ We can identify $P \longleftrightarrow L(P)$



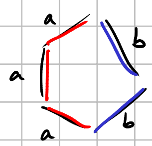
Example



are not balanced

Consequence of (C2) : There is a bijection $A \longleftrightarrow$ atoms of P

(C3) Weakly graded There is a "norm" $\nu : A \to \mathbb{N}_+$ such that any two maximal chains in some $[x,y]$ have the same total norm



$\nu(a) = 2$ $\nu(b) = 3$

$\Rightarrow$ Every language in $L(P)$ has a well-defined norm and also every element of M

(C4) Lattice

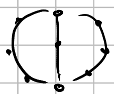
Def If P satisfies (C1) – (C4) then it is a combinatorial Garside structure, M is a Garside monoid, G is a Garside group

Examples

- $P = B_n \qquad M = \mathbb{N}^n \qquad G = \mathbb{Z}^n$
- $P = (\mathfrak{S}_n$, weak order$) \qquad M =$ braid monoid $\qquad G =$ braid group on n strands
- $P = NC_n \qquad M =$ dual braid monoid $\qquad G \cong$ ↑

Exercise: check (C1)–(C4)

- $G = \langle a_1, \dots, a_n \mid a_1^{m_1} = a_2^{m_2} = \dots = a_n^{m_n} \rangle$

  $P =$ 

- $\langle a_1, \dots, a_n \mid a_1 a_2 \cdots a_k = a_2 a_3 \cdots a_{k+1} = \dots = a_n a_1 a_2 \cdots a_{n-1} \rangle$

...

Identify $P \cong L(P)$ so that $P \cong L(P) \longrightarrow M \longrightarrow G$

$x \mapsto \mathcal{L}(1, x)$

If all are injections, then P is an interval in the right Cayley graph of G.

Garside monoids

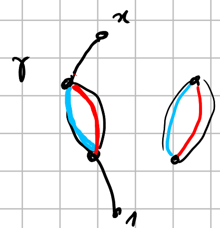
Lemma $P \to M$ is injective

Proof Let $a_1 \cdots a_k \in \mathcal{L}(1, x)$ be a word that labels a chain γ in $[1, x]$ $(x \in P)$.

$$a_1 \cdots a_{i-1} \underbrace{a_i \cdots a_j}_{\downarrow} a_{j+1} \cdots a_k$$

$$b_1 \cdots b_m$$

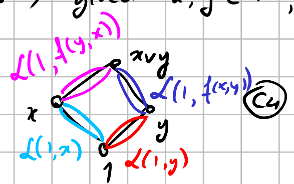
some relation of M



By (C1), $a_i \cdots a_j$ determines its language

$\Rightarrow a_1 \cdots a_{i-1} b_1 \cdots b_m a_{j+1} \cdots a_k$ labels a chain from 1 to x $\longrightarrow$ gives the same element x of P $\square$

To define M: we can use $\overline{P} = P \setminus \{1\}$ as generators; given $x, y \in \overline{P}$,



$f: P \times P \to 1$

$f(x,y) \in P$

$\mathcal{L}(x, x \vee y) = \mathcal{L}(1, f(y,x))$

$\mathcal{L}(y, x \vee y) = \mathcal{L}(1, f(x,y))$

$$M = \langle \overline{P} \mid x\, f(y,x) = y\, f(x,y) \quad \forall\, x, y \in \overline{P} \rangle$$

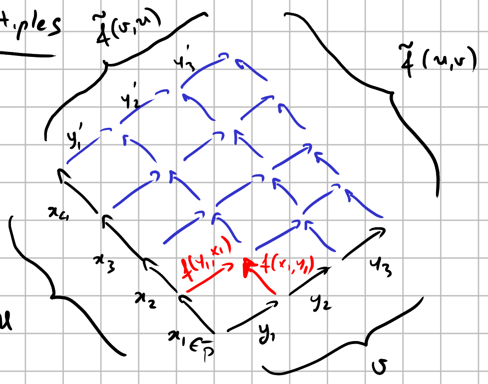
And these relations are enough because $x \vee y$ is the join (Exercise)

if $f(x,y) = 1$, then we put the empty word ε instead of $f(x,y)$

$\Rightarrow$ Elements of M are represented by words in $\overline{P}$ and two words $u, v \in \overline{P}^*$ are equivalent $(u \equiv v)$ if they define the same element of M.

Right common multiples

Let $u, v \in \overline{P}^*$



This defines $\tilde{f} : \overline{P}^* \times \overline{P}^* \longrightarrow \overline{P}^*$ extending f

By construction, $\boxed{u\,\tilde{f}(v,u) \equiv v\,\tilde{f}(u,v)}$

Lemma 1 Given $u, v, u', v' \in \overline{P}^*$, the following are equivalent:

(i) $uu' \equiv vv'$

(ii) $u' \equiv \tilde{f}(v,u)\,w$ and $v' \equiv \tilde{f}(u,v)\,w$ for some $w \in \overline{P}^*$.

Proof (ii) $\Rightarrow$ (i) obvious

(i) $\Rightarrow$ (ii) a big induction [Dehornoy , Lemma 4]

Lemma 2 (a) The class of u in M left-divides the class of v iff $\tilde{f}(u,v) = \varepsilon$. (empty word)

(b) $u \equiv v$ iff $\tilde{f}(u,v) = \tilde{f}(v,u) = \varepsilon$ $\rightsquigarrow$ we can check equality ($\equiv$) in M!

In particular, left divisibility in M is a partial order relation (antisymmetric)

Proof (a) $u\,\tilde{f}(v,u) \equiv v\,\tilde{f}(u,v)$ by the property of $\tilde{f}$

- If $\tilde{f}(u,v) = \varepsilon$, then u left-divides v
- If u left-divides v, then $v\varepsilon \equiv uu' \overset{\text{(Lemma 1)}}{\Longrightarrow} \varepsilon \equiv \tilde{f}(u,v)\cdot w$

$\underset{C3}{\Longrightarrow} \tilde{f}(u,v) = \varepsilon$

(b) • "$\Rightarrow$" follows from (a)

• "$\Leftarrow$" if $\tilde{f}(u,v) = \tilde{f}(v,u) = \varepsilon$, then $u = u\underbrace{\tilde{f}(v,u)}_{\varepsilon} \equiv v\underbrace{\tilde{f}(u,v)}_{\varepsilon} = v$ $\square$

Corollary 1 $v\,\tilde{f}(u,v) \equiv u\,\tilde{f}(v,u)$ is the right lcm of the classes of u and v in M.

$\Rightarrow$ left gcd exist ($[1, u \vee v]$ is a finite join-semilattice ($\leq$ is left divisibility) $\searrow$ lattice)

Corollary 2 M is left-cancellative: $uv \equiv u'v'$ and $u \equiv u' \Rightarrow v \equiv v'$.

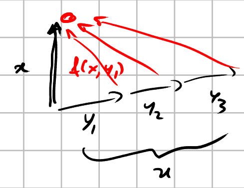
Proof $uv \equiv u'v' \overset{\text{Lemma 1}}{\Longrightarrow} \begin{cases} v \equiv \tilde{f}(u',u)\,w = w \\ v' \equiv \tilde{f}(u,u')\,w = w \end{cases} \Rightarrow v \equiv v'$ $\square$

Lemma 2

Corollary 3 (a) If the class of u left-divides some element $x \in P$, then it is in P

(b) $P \subseteq M$ is the set of left divisors of δ.

Proof (a) We have $\tilde{f}(u,x) = \varepsilon$ by lemma 2.



By induction on $|u|$, this is a diagram in P

$\Rightarrow u \in P$.

(b) It is enough to show that every $x \in P$ left-divides δ in M (then we conclude with part (a))

This is obvious: $\tilde{f}(x,\delta) = f(x,\delta) \underset{\substack{x \leq \delta \\ \text{in } P}}{=} \varepsilon \overset{\text{Lemma 2}}{\Longrightarrow} x$ left-divides δ in M. □