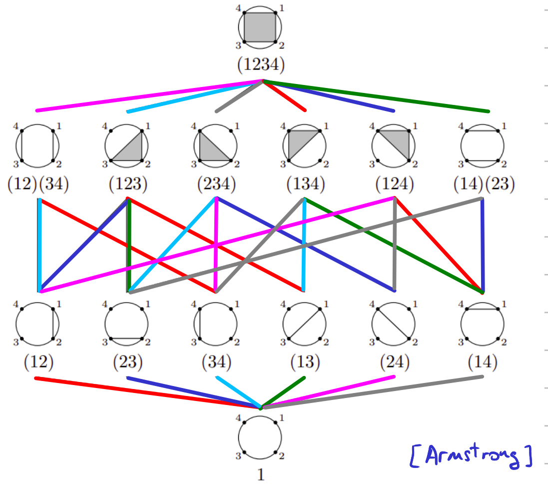
Combinatorial Topology and Group Theory

Lecture V
07/12/2021

Example: Why is NC_n balanced?



[Armstrong]

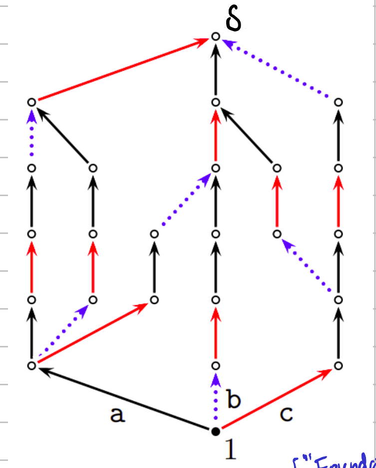
$(14)(23)(13) = (1234)$

$(24)(14)(23) = (1234)$

(24) = conjugate of (13) wrt $(14)(23)$

The generating set is closed under conjugation!

Example: A combinatorial Garside structure which is not self-dual ($P \not\cong P^*$).



["Foundations"]

Everything works if we swap "left" and "right". (The interval between 1 and δ in the left Cayley graph of G or M will be P^*)

Theorem Let M ($\to M(P)$) be a Garside monoid.

weaker than (C3) → • M is "atomic" (for every $u \in M$, the length of every expression $u = x_1 \cdots x_k$ ($x_i \in A$) is bounded by $\nu(u)$)

- M is left- and right-cancellative
- M has left and right gcd's and lcm's
- There is $\delta \in M$ ("Garside element") such that $\{\text{left divisors of } \delta\} = \{\text{right divisors of } \delta\}$ is finite, generates M (C2)

Remark This is often taken as the definition of Garside monoid.

Remark δ Garside element $\Rightarrow$ δ^k Garside element

——— Let's write $=$ instead of $\equiv$ for equality in M ———

Proposition There is a permutation $\xi : P \to P$ satisfying $\delta \cdot x = \xi(x) \cdot \delta$ for all $x \in P$.
In addition, $\xi|_A : A \to A$ is a permutation of the atoms.

In particular, if n is the order of ξ, then δ^n is in the center of M.

Proof Given $x \in P$, there is $x' \in P$ such that $x'x = \delta$ and there is $\xi(x)$ such that $\xi(x)\, x' = \delta$.

$$\Rightarrow \quad \delta x = \xi(x)\, x' x = \xi(x)\, \delta .$$

ξ is well-defined & injective by cancellativity

If $x = yz$, then $\xi(yz)\,\delta = \xi(x)\,\delta = \delta x = \delta y z = \xi(y)\,\xi(z)\,\delta \Rightarrow \xi(yz) = \xi(y)\xi(z)$

$\Rightarrow$ ξ sends non-atoms to non-atoms
and atoms to atoms.

$\square$

Example Braid monoid ($P = (\mathfrak{S}_n$, weak order$)$), generators $s_i = (i\ \ i+1)$ $\quad \delta = \begin{matrix} n & n-1 & \cdots & 2 & 1 \end{matrix}$ $(1\,n)(2\ n-1)\cdots$

$\delta s_i = \xi(s_i)\,\delta$ has to hold also in $\mathfrak{S}_n$

$\Rightarrow$ $\xi(s_i) = \delta s_i \delta^{-1} = s_{n-i}$

$\Rightarrow$ ξ has order 2

$\Rightarrow$ $\delta^2 \in$ center of M
(in fact, it generates the center of M)
↖ Exercise

Left-greedy normal form (in M)

We say that $u = x_1 x_2 \cdots x_k$ is in normal form if $x_i = \mathrm{lgcd}(\delta, x_i x_{i+1} \cdots x_k)$ and $x_i \in \overline{P}$.

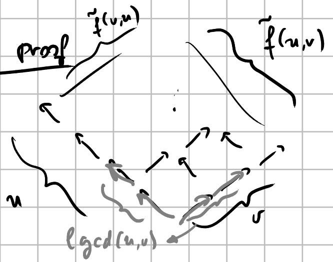
(this defines the normal form uniquely)

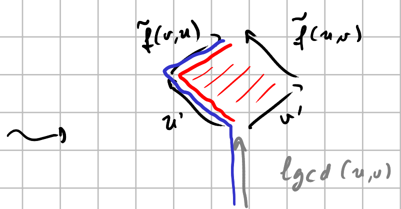
$x_1 = \mathrm{lgcd}(\delta, u)$

$u = x_1 \cdot u'$ and u' is unique by cancellativity

Lemma $\mathrm{rlcm}(u,v) = \mathrm{lgcd}(u,v) \cdot \mathrm{llcm}(\tilde{\xi}(u,v), \tilde{\xi}(v,u))$ for all $u, v \in M$.

Sketch of proof



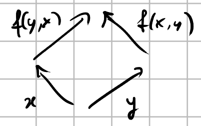


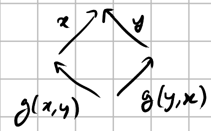
$u = \mathrm{lgcd}(u,v)\ u'$
$v = \mathrm{lgcd}(u,v)\ v'$

(Exercise) $\square$

$$\Rightarrow \quad \mathrm{lgcd}(u,v) = \tilde{g}\Big(\mathrm{llcm}\big(\tilde{f}(u,v), \tilde{f}(v,u)\big), \quad \mathrm{rlcm}(u,v) \Big)$$

$\tilde{g}$: same as $\tilde{f}$ but to compute left common multiples





+ extend g to $\tilde{g}$

Garside groups

Theorem (a) The natural map $M \to G$ is injective
(b) Every element of G can be written as uv^{-1} (resp. $u^{-1}v$) for some $u,v \in M$
(c) u,v are unique if $\mathrm{rgcd}(u,v) = 1$ (resp. $\mathrm{lgcd}(u,v) = 1$)

Proof (a) and (b) follow from Ore's theorem (M left and right cancellative
+ $\exists$ right common multiples
$\Rightarrow M \hookrightarrow G$)

[Clifford–Preston, Theorem 1.23]

(c) is obvious.

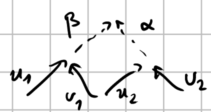
Idea of (a) and (b): construct a "group of fractions" $\tilde{G}$ of M, which will be G.

$$\tilde{G} = \{(u,v) \in M \times M\}$$

uv^{-1}

$(u,v) \sim (u',v')$ whenever $u\beta = u'\beta'$
where β, β' are chosen such that $v\beta = v'\beta'$.

Group operation:

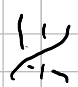


$$(u_1,v_1)(u_2,v_2) := (u_1\beta, \ v_2\alpha) \quad \text{where} \quad v_1\beta = u_2\alpha$$

$M \hookrightarrow \tilde{G}$ and $G \cong \tilde{G}$
$u \mapsto (u,1)$ $\quad a \mapsto (a,1)$ for all atoms a
$uv^{-1} \leftarrow (u,v)$

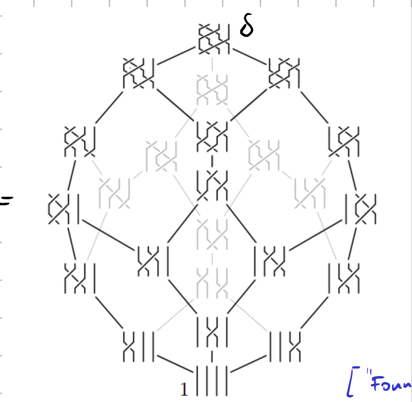
$\square$

Intermezzo on positive braids

$\delta =$  "half twist"

$= s_1 s_2 s_1 s_3 s_2 s_1 \cdots$

$s_i \delta = \delta s_{n-i}$

$P =$ 

["Foundations"]

Normal form in G

Theorem For every $g \in G$, there is $n \in \mathbb{N}$ such that $\delta^n g \in M$ (same for $g\delta^n$)

and g can be written uniquely as $g = \delta^m x_1 \cdots x_k$ for $m \in \mathbb{Z}$, $x_1 \cdots x_k$ normal form in M $(x_i \in \overline{P})$ and $x_1 \neq \delta$.

Proof Let $g = y_1^{\pm 1} \cdots y_k^{\pm 1}$, $y_i \in \overline{P}$. Then $\delta^n g \in M$ where $n = \#$ "-1" (induction):

$$\delta^n g = \begin{cases} \xi^n(y_1)\,\delta^n\, y_2^{\pm 1} \cdots y_k^{\pm 1} = \dots & \text{if } y_1^{+1} \\ \delta^{n-1} \cdot \underbrace{\delta y_1^{-1}}_{x_1 \in P} \; y_2^{\pm 1} \cdots y_k^{\pm 1} = \dots & \text{if } y_1^{-1} \end{cases}$$

Define m as the largest integer such that $\delta^{-m} g \in M$. $\square$

Classifying space of G

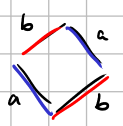
Let $K = \Delta(P)/\approx$ where two chains $\left| \begin{array}{c} y_0 < y_1 < \cdots < y_k \\ \downarrow \quad \downarrow \qquad \downarrow \\ y_0' < y_1' < \cdots < y_k' \end{array} \right.$,

are identified iff $\mathcal{L}(y_i, y_{i+1}) = \mathcal{L}(y_i', y_{i+1}') =: \mathcal{L}(1, x_{i+1})$

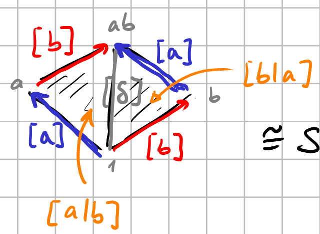
define $x_0, \dots, x_k$

$\Rightarrow$ The cells of K are indexed by tuples $[x_1 | x_2 | \cdots | x_k]$ such that $x_i \in \overline{P}$ and $x_1 x_2 \cdots x_k \in P$.

Examples

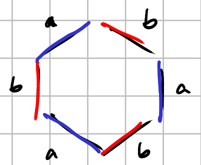
- $P =$ 

$\delta = ab = ba$

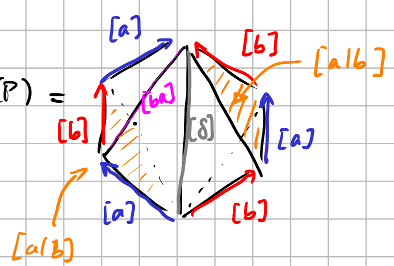
$K = \Delta(P)/\approx \; =$ 

$\cong S^1 \times S^1$

$G = \mathbb{Z} \times \mathbb{Z}$

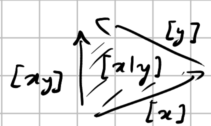
- $P =$ 

$\delta = aba = bab$

$\Delta(P) =$ 

K has one 0-cell $[\,]$, and a 1-cell $[x]$ for each $x \in \overline{P}$

Lemma $\pi_1(K, [\,]) \cong G$ [Hatcher, Proposition 1.26]



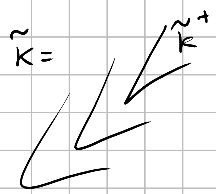
Theorem $\tilde{K}$ is contractible, so $K \simeq K(G,1)$.

Proof $\tilde{K}$ is naturally a simplicial complex with simplices $[g \| x_1 | x_2 | \dots | x_k]$ for $g \in G$, $[x_1 | \dots | x_k] \in K$

with vertices

$[g]$, $[g x_1]$, $\dots$, $[g x_1 \cdots x_k]$

$$\left(\begin{array}{l} G \text{ acts on } \tilde{K} : \; [g \| x_1 | \dots | x_k] \xmapsto{h} [hg \| x_1 | \dots | x_k] \\ \Rightarrow \tilde{K} \longrightarrow K = \tilde{K}/G \text{ is a covering map} \end{array} \right)$$

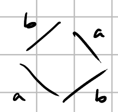
$\tilde{K} =$ 

Let $\tilde{K}^+$ be the subcomplex consisting of $[u \| x_1 | \dots | x_k]$ with $u \in M$.

Then $\tilde{K} = \bigcup_{n \in \mathbb{N}} \delta^{-n} \tilde{K}^+$, so it is enough to prove that $\tilde{K}^+$ is contractible (Whitehead)

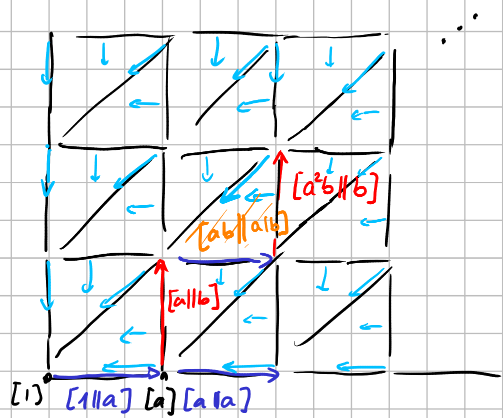
We use discrete Morse theory:

$$M = \left\{ [u x_1 \| x_2 | \dots | x_k] <\cdot [u \| x_1 | \dots | x_k] \;\middle|\; \operatorname{rgcd}(\delta, u x_1 \cdots x_k) = x_1 \cdots x_k \right\}$$

Example 

$K \cong S^1 \times S^1$

$\tilde{K} \cong \mathbb{R} \times \mathbb{R}$

$\tilde{K}^+ = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ 

M is compatible with the poset map $\theta : \mathcal{F}(\tilde{K}) \longrightarrow (M, \leq)$ ← left divisibility

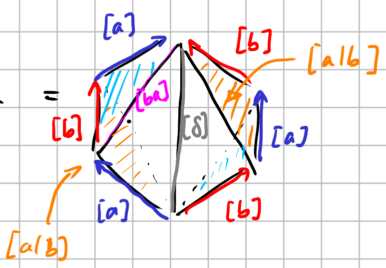
$[u \| x_1 | \dots | x_k] \longmapsto u x_1 \cdots x_k$

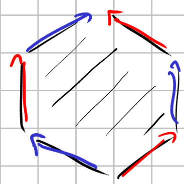
- $\theta^{-1}(M_{\leq v})$ is finite
  $\Rightarrow$ M is proper
- Acyclicity of M can be checked on the fibers $\theta^{-1}(w)$ □

Corollary
- $H_*(G) = H_*(K)$
- homological dimension of $G \leq \ell(P)$
- G is torsion-free

Open question: can we relax the lattice property and get that $K \simeq K(G,1)$? ← C4

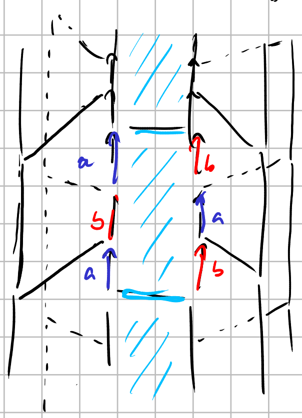
Example $\langle a, b \mid aba = bab \rangle$

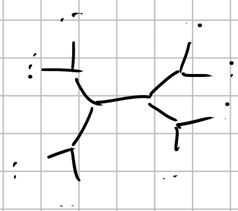
$K =$ 

$\simeq$ 

Salvetti complex

universal cover



$\cong$  $\times \mathbb{R}$