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Discrete Morse theory and the $K(\pi, 1)$ conjecture

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Contents

Introduction

The aim of this thesis is to present the $K(\pi,1)$ conjecture for Artin groups, an open conjecture which goes back to the 70s, and to use the technique of discrete Morse theory to prove some related results.

The beginning of the study of Artin groups dates back to the introduction of braid groups in the 20s. Artin groups where defined in general by Tits and Brieskorn in the 60s, in relation to the theory of Coxeter groups and singularity theory. Deep connections with the main areas of mathematics where discovered: in addition to the theory of Coxeter groups and singularity theory, Artin groups naturally arise in the study of root systems, hyperplane arrangements, configuration spaces, combinatorics, geometric group theory, knot theory, mapping class groups and moduli spaces of curves.

The study of Artin groups deeply relies on the study of Coxeter groups, i.e. groups with a presentation of the form

$$W = \big\langle s_1, \dots, s_n \mid s_i^2 = 1,\, (s_i s_j)^{m_{ij}} = 1 \big\rangle.$$

Despite their purely algebraic definition, Coxeter groups admit an interesting geometric interpretation: each of them can be embedded as a subgroup of $\mathrm{GL}_n(\mathbb{R})$ generated by n reflections with respect to hyperplanes of $\mathbb{R}^n$. Artin groups on the other way are defined through a representation of a very similar form:

$$A = \big\langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \dots}_{m_{ij}} = \underbrace{s_j s_i s_j \dots}_{m_{ij}} \big\rangle.$$

As one can see, the presentation of a Coxeter group W can be obtained from the presentation of the corresponding Artin group A by adding the relation $s_i^2 = 1$ for each generator s_i. Differently from Coxeter groups, that are sometimes finite, Artin groups are always infinite.

There are many properties conjectured to be true for all Artin groups but proved only for some families of them, e.g. being torsion-free, having a trivial center, and having solvable word problem. Some of these problems, and also others (such as the computation of homology and cohomology), are related to an important conjecture called "$K(\pi,1)$ conjecture". Such conjecture says that a certain topological space $\bar{N}$, constructed from a Coxeter group using the representation we mentioned above, is a classifying space for the corresponding

Artin group. The space $\bar{N}$ admits finite CW models, therefore the $K(\pi, 1)$ conjecture directly implies that Artin groups are torsion-free.

A tool which is very important in our work is discrete Morse theory, introduced by Forman in the 90s. Discrete Morse theory allows to prove the homotopy equivalence of CW complexes through elementary collapses of cells, on the basis of some combinatorial rules which can be naturally expressed with the language of graph theory. The idea of using discrete Morse theory to prove results about the $K(\pi, 1)$ conjecture is present in the literature only in very recent works.

This thesis is structured as follows. In the first chapter we present some of the most important known results about Coxeter groups, especially concerning their geometric and combinatorial properties. In the second chapter we do the same for Artin groups. In particular we introduce the Artin monoids, which are significantly important in the study of Artin groups. The third chapter is devoted to an introduction to the terminology and the main results of discrete Morse theory, in a version developed by Chari and Batzies after the original work of Forman. In the fourth chapter we introduce the $K(\pi, 1)$ conjecture together with some of its consequences. We define a particular CW model for the space $\bar{N}$, called Salvetti complex, and we describe its combinatorial structure. Then we give a new proof of the $K(\pi, 1)$ conjecture for Artin groups of finite type (i.e. those for which the corresponding Coxeter group is finite), using discrete Morse theory. Finally, in the fifth chapter we describe some connections between the $K(\pi, 1)$ conjecture and classifying space of Artin monoids. A relevant result in this direction is a theorem by Dobrinskaya published in 2006, which states that the classifying space of an Artin monoid is homotopy equivalent to the corresponding space $\bar{N}$ mentioned above. We prove that applying discrete Morse theory one can collapse the standard CW model for the classifying space of an Artin monoid and obtain the Salvetti complex. In particular, this gives an alternative prove of Dobrinskaya's theorem.

Chapter 1

Coxeter groups

In this chapter we will introduce Coxeter groups and we will present some of their algebraic, combinatorial and geometric properties. The main references are [Bou68, Dav08, Hum92].

1.1 Coxeter systems

Let S be a finite set, and let $M = (m_{s,t})_{s,t \in S}$ be a square matrix indexed by S and satisfying the following properties:

- $m_{s,t} \in \{2, 3, \dots\} \cup \{\infty\}$ for all $s \neq t$, and $m_{s,t} = 1$ for $s = t$;
- $m_{s,t} = m_{t,s}$ i.e. M is symmetric.

Such a matrix is called a *Coxeter matrix*. From a Coxeter matrix M we can construct a non-oriented simple edge-labelled graph Γ, called *Coxeter graph* of M, as follows:

- we take S as the set of vertices;
- an edge connects vertices s and t if and only if $m_{s,t} \geq 3$, and such edge is labelled by $m_{s,t}$.

For the sake of conciseness, when $m_{s,t} = 3$ the label on the corresponding edge is traditionally omitted. Thus an edge connecting vertices s and t is labelled only if $m_{s,t} \geq 4$. Sometimes we also say that (Γ, S) is a Coxeter graph, if we want to underline that S is the set of vertices of Γ.

Definition 1.1. Let (Γ, S) be a Coxeter graph. The *Coxeter system* of (Γ, S) is the pair (W_Γ, S), where W_Γ is the group defined by

$$W_\Gamma = \langle\, S \mid (st)^{m_{s,t}} = 1 \;\; \forall\, s, t \in S \text{ such that } m_{s,t} \neq \infty \,\rangle.$$

A group W_Γ as above is called *Coxeter group*.

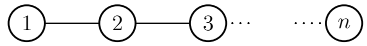


Figure 1.1: Coxeter graph of type A_n.

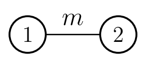


Figure 1.2: Coxeter graph of type $I_2(m)$.

The notion of Coxeter group includes many families of important groups. For instance setting $m_{s,t} = 2$ for all $s \neq t$ one obtains the direct product of $|S|$ copies of $\mathbb{Z}_2$. In a similar way, setting $m_{s,t} = \infty$ for all $s \neq t$ one obtains the free product of $|S|$ copies of $\mathbb{Z}_2$.

A less trivial example is given by the family of Coxeter graphs shown in Figure 1.1. Such graphs (and the corresponding Coxeter groups) are said to be of type A_n, where n is the size of S. If we call $s_1, \dots, s_n$ the elements of S, the relations in the Coxeter group of type A_n are given by:

- $s_i^2 = 1$ for all i;
- $(s_i s_{i+1})^3 = 1$ for $i = 1, \dots, n-1$;
- $(s_i s_j)^2 = 1$ for all i, j such that $|i - j| \geq 2$.

Such group turns out to be isomorphic to $\mathfrak{S}_{n+1}$, the symmetric group on $n+1$ elements. Indeed, the generator s_i corresponds to the transposition $(i \ \ i+1)$ in $\mathfrak{S}_{n+1}$.

As another example, if S consists only of two elements s, t with $m = m_{s,t} \neq \infty$, the obtained Coxeter group is the dihedral group $\mathfrak{D}_m$ on $2m$ elements (the group of symmetries of a regular m-agon). Indeed, s and t can be regarded as linear reflections in $\mathbb{R}^2$ with respect to lines forming an angle of π/m with each other, and st is then a rotation of $2\pi/m$. The corresponding Coxeter graph is said to be of type $I_2(m)$, and is displayed in Figure 1.2.

The following lemma shows that from a Coxeter system (W_Γ, S) one can recover all the information encoded in its Coxeter graph Γ. The proof will be a direct consequence of Proposition 1.10.

Lemma 1.2 ([Bou68])**.** Let (W_Γ, S) be a Coxeter system. Then, for any $s, t \in S$, the order of the element st in W_Γ is precisely $m_{s,t}$. In particular every $s \in S$ has order 2, and st has infinite order whenever $m_{s,t} = \infty$.

In view of Lemma 1.2 we will often write W instead of W_Γ to indicate a Coxeter group.

Definition 1.3. Let (W, S) be a Coxeter system. For any $T \subseteq S$ let W_T be the subgroup of W generated by the elements of T. A subgroup constructed in this way is called a *standard parabolic subgroup* of W.

The standard parabolic subgroups of a Coxeter group W are Coxeter groups themselves, as we are going to state in the following lemma. The proof will be given in Section 1.3.

Lemma 1.4 ([Hum92], Theorem 5.5)**.** Let (W, S) be the Coxeter system corresponding to the Coxeter graph (Γ, S), and let T be a subset of S. Then the pair (W_T, T) is also a Coxeter system, with associated Coxeter graph $\Gamma|_T$.

Definition 1.5. A Coxeter system (W, S) is *irreducible* if the corresponding Coxeter graph Γ is connected.

Two generators $s, t \in S$ commute if they belong to different connected components of the Coxeter graph Γ. Thus a Coxeter group W is isomorphic to the direct product of the parabolic subgroups $W_{T_1}, \dots, W_{T_k}$ corresponding to the connected components $\Gamma|_{T_1}, \dots, \Gamma|_{T_k}$ of Γ. For this reason the study of a Coxeter group can be essentially reduced to the study of its maximal irreducible parabolic subgroups.

We finally introduce the length function of a Coxeter system (W, S), which is a very important concept in the study of Coxeter groups.

Definition 1.6. Fix a Coxeter system (W, S), and let $w \in W$. An *expression* for w is an element $s_1 \cdots s_k$ of the free monoid on S, such that the equality $w = s_1 \cdots s_k$ holds in W. An expression for w is said to be *reduced* if it has minimal length among all the expressions for w.

Remark 1.7. The map $W \to \mathbb{Z}_2$ which sends an element $w \in W$ to the parity of the length of any (not necessarily reduced) expression for w is a well defined group homomorphism. This is true since the relations that define a Coxeter group always have even length.

Definition 1.8. Let (W, S) be a Coxeter system. The *length function* $\ell \colon W \to \mathbb{N}$ is defined as follows: for any $w \in W$, $\ell(w)$ is length of any reduced expression for w.

Many interesting properties of the length function will be investigated later in this chapter. However, we point out some of the simplest ones:

- $\ell(1) = 0$, and $\ell(w) = 1$ if and only if $w \in S$;
- $\ell(w^{-1}) = \ell(w)$ for all $w \in W$, since if $w = s_1 \cdots s_k$ then $w^{-1} = s_k \cdots s_1$ and vice versa;
- $\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w')$ for all $w, w' \in W$;
- $\ell(ws) = \ell(w) \pm 1$ for all $w \in W$ and $s \in S$, since $\ell(ws)$ and $\ell(w)$ have different parity (by Remark 1.7) and differ at most by 1;
- if $w \neq 1$ then there exists some $s \in S$ such that $\ell(ws) = \ell(w) - 1$ (choose s as the last generator of any reduced expression for w).

If we have a standard parabolic subgroup W_T of W, then we also have a length function $\ell_T \colon W_T \to \mathbb{N}$ which sends each element w of W_T to the minimum length of an expression of w as product of generators in T. The following lemma will be also proved in Section 1.3. For now we simply notice that $\ell(w) \leq \ell_T(w)$ for all $w \in W_T$.

Lemma 1.9 ([Hum92], Theorem 5.5)**.** Let (W,S) be a Coxeter system. Then for all $T \subseteq S$ the length function ℓ_T coincides with the restriction $\ell|_{W_T}$.

1.2 Geometric representation

Coxeter groups have a strong geometric meaning. As we are going to see, they admit a faithful representation as groups generated by (non necessarily orthogonal) reflections in some real vector space V. As a reflection we mean a linear endomorphism of V that pointwise fixes a hyperplane and sends some nonzero vector to its negative.

We have already seen in Section 1.1 that dihedral groups, i.e. Coxeter groups of type $I_2(m)$, can be represented as subgroups of $\mathrm{GL}(2,\mathbb{R})$ generated by two reflections. In the case of the symmetric group $\mathfrak{S}_{n+1}$ it is also easy to construct a faithful representation: if $s_1, \dots, s_n$ are the transpositions defined in Section 1.1, one can regard s_i as the orthogonal reflection with respect to the hyperplane $\{x_i = x_{i+1}\}$ in $\mathbb{R}^{n+1}$. In this way one obtains the standard action of $\mathfrak{S}_{n+1}$ on $\mathbb{R}^{n+1}$ by permutation of the coordinates. If we restrict such representation to the hyperplane $\{x_1 + \dots + x_{n+1} = 0\}$ of $\mathbb{R}^{n+1}$ we get a faithful representation of $\mathfrak{S}_{n+1}$ of dimension n.

Let's move to the general case. Given a Coxeter system (W,S), consider a real vector space V of dimension $|S|$ having as a basis the set $\{e_s \mid s \in S\}$. Define on V a symmetric bilinear form B as follows:

$$B(e_s, e_t) = -\cos \tfrac{\pi}{m_{s,t}},$$

where $\frac{\pi}{m_{s,t}}$ is 0 whenever $m_{s,t} = \infty$. Notice that $B(e_s, e_s) = 1$ for all $s \in S$, and that in particular all the vectors e_s are non-isotropic. For each $s \in S$, define a linear transformation $\rho_s \colon V \to V$ in the following way:

$$\rho_s(v) = v - 2B(e_s, v)\, e_s.$$

The endomorphism ρ_s is a reflection since it sends e_s to $-e_s$ and pointwise fixes the hyperplane orthogonal to e_s with respect to B. In particular, ρ_s has order 2. Notice also that ρ_s preserves B for all $s \in S$: for all $v, w \in V$ we have

$$\begin{aligned} B\big(\rho_s(v), \rho_s(w)\big) &= B\big(v - 2B(e_s, v)\, e_s, w - 2B(e_s, w)\, e_s\big) \\ &= B(v,w) - 2B(e_s,v)B(e_s,w) - 2B(e_s,w)B(e_s,v) \\ &\quad + 4B(e_s,v)B(e_s,w)B(e_s,e_s) \\ &= B(v,w). \end{aligned}$$

Proposition 1.10 (cf. [Hum92], Proposition 5.3)**.** The order of $\rho_s\rho_t$ in $\mathrm{GL}(V)$ is $m_{s,t}$ for all $s, t \in S$.

Proof. As we already noticed, ρ_s is a reflection for all s and therefore has order 2. So the proposition holds for $s = t$.

For $s \neq t$ set $m = m_{s,t}$ and consider the linear subspace $V_{s,t}$ of V generated by e_s and e_t. Both the reflections ρ_s and ρ_t fix $V_{s,t}$, and so does the composition $\rho_s\rho_t$. Call $\bar\rho_s$ and $\bar\rho_t$ the restrictions to $V_{s,t}$ of ρ_s and ρ_t, respectively. If $m = \infty$ the matrix associated to $\bar\rho_s\bar\rho_t$ with respect to the basis $\{e_s, e_t\}$ of $V_{s,t}$ is

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix},$$

which has characteristic polynomial and minimal polynomial both equal to $(t-1)^2$. Therefore in this case $\bar\rho_s\bar\rho_t$ is not diagonalizable over $\mathbb{C}$, and in particular has infinite order. Then the same conclusion holds for $\rho_s\rho_t$.

Assume from now on that $m \neq \infty$. The restriction of B to $V_{s,t}$ is represented in coordinates with respect to the basis $\{e_s, e_t\}$ by the following matrix:

$$M = \begin{pmatrix} 1 & -\cos\frac{\pi}{m} \\ -\cos\frac{\pi}{m} & 1 \end{pmatrix}.$$

Such matrix is positive definite by Sylvester's criterion since $\det M > 0$ for $m \neq \infty$. Then there exists a linear isomorphism $\psi\colon V_{s,t} \to \mathbb{R}^2$ that sends B to the standard scalar product of $\mathbb{R}^2$. The reflections $\rho'_s = \psi \circ \bar\rho_s \circ \psi^{-1}$ and $\rho'_t = \psi \circ \bar\rho_t \circ \psi^{-1}$ are reflections in $\mathbb{R}^2$ that preserve the standard scalar product, i.e. orthogonal reflections. The vectors e_s and e_t are sent to $e'_s = \psi(e_s)$ and $e'_t = \psi(e_t)$ such that

$$\langle e'_s, e'_t\rangle = B(e_s, e_t) = -\cos\tfrac{\pi}{m}.$$

Then the lines spanned by e'_s and e'_t form an angle of $\frac{\pi}{m}$ with each other. Since ρ'_s and ρ'_t are precisely the orthogonal reflections with respect to e'_s and e'_t, the composition $\rho'_s\rho'_t$ is a rotation of $\frac{2\pi}{m}$. So $\rho'_s\rho'_t$ has order m, and also does $\bar\rho_s\bar\rho_t$. Since B is non-degenerate on $V_{s,t}$, the entire space V is the direct sum of $V_{s,t}$ and its orthogonal $V_{s,t}^{\perp}$ with respect to B. It is easy to check that ρ_s and ρ_t are both the identity on $V_{s,t}^{\perp}$, so $\rho_s\rho_t$ has order m. $\square$

An immediate consequence of the previous proposition is that there exists a group homomorphism

$$\rho\colon W \to \mathrm{GL}(V)$$

which sends s to ρ_s for all $s \in S$. Such homomorphism is called the *canonical representation* of (W, S). As anticipated the canonical representation of a Coxeter system is always faithful, but we will need some more considerations to prove it. Meanwhile, we can use the existence of the canonical representation to prove Lemma 1.2.

Proof of Lemma 1.2. Since $\rho_s = \rho(s)$ has order 2 for any $s \in S$, s itself must have order multiple of 2. Since $s^2 = 1$, the order of s is exactly 2.

Consider now elements $s \neq t$ in S. By Proposition 1.10 the order of $\rho_s\rho_t = \rho(st)$ is $m_{s,t}$, so the order of st must be a multiple of $m_{s,t}$. If $m_{s,t} = \infty$ this means that the order of st is also ∞. If $m_{s,t} \neq \infty$ we know that $(st)^{m_{s,t}} = 1$, so the order of st is exactly $m_{s,t}$. □

To simplify the notation, in the rest of this chapter we will write $w(v)$ instead of $\rho(w)(v)$, for $w \in W$ and $v \in V$.

1.3 Roots

In this section we are going to further investigate the geometry of the canonical representation of a Coxeter system, which is connected with more combinatorial properties such as the behaviour of the length function.

Let (W, S) be a Coxeter system and $\rho \colon W \to \mathrm{GL}(V)$ its canonical representation. Recall that the standard basis of V consists of the vectors e_s for $s \in S$.

Definition 1.11. The *root system* Φ of (W, S) is the subset of V given by

$$\Phi = \{w(e_s) \mid w \in W,\, s \in S\}.$$

The elements of Φ are called *roots*.

All the roots of (W, S) are unit vectors with respect to the bilinear form $B \colon V \times V \to \mathbb{R}$ defined in Section 1.2, since s preserves B for all $s \in S$ and thus w preserves B for all $w \in W$. Moreover, since $s(e_s) = -e_s$, the opposite of any root is also a root.

Let $\zeta \in \Phi$ be a root. Then it can be written uniquely in the form

$$\zeta = \sum_{s \in S} c_s e_s,$$

for some $c_s \in \mathbb{R}$. The root ζ is said to be *positive* if $c_s \geq 0$ for all $s \in S$, and *negative* if $c_s \leq 0$ for all $s \in S$. In the former case we write $\zeta > 0$ whereas in the latter case we write $\zeta < 0$. Denote by Φ^+ the set of positive roots and by Φ^- the set of negative roots.

Theorem 1.12 ([Hum92], Theorem 5.4)**.** Let $w \in W$ and $s \in S$. If $\ell(ws) > \ell(w)$, then $w(e_s) > 0$. If $\ell(ws) < \ell(w)$, then $w(e_s) < 0$.

Proof. We prove by induction on $\ell(w)$ that if $\ell(ws) > \ell(w)$ then $w(e_s) > 0$. If $\ell(w) = 0$ then $w = 1$, and the claim is trivial. Suppose then $\ell(w) > 0$, and choose $t \in S$ such that $\ell(wt) = \ell(w) - 1$. Notice that $t \neq s$, and set $T = \{s, t\} \subseteq S$. Consider now the subset C of the coset wW_T defined by

$$C = \{u \in wW_T \mid \ell(u) + \ell_T(u^{-1}w) = \ell(w)\}.$$

The set C is non-empty because clearly $w \in C$. Choose $u \in C$ such that $\ell(u)$ is minimized, and set $u_T = u^{-1}w \in W_T$.

Assume by contradiction that $\ell(us) = \ell(u) - 1$. Then

$$\begin{aligned}
\ell(w) &\leq \ell(us) + \ell((us)^{-1}w) \\
&= \ell(us) + \ell(su^{-1}w) \\
&\leq \ell(us) + \ell_T(su^{-1}w) \\
&= \ell(u) - 1 + \ell_T(su^{-1}w) \\
&\leq \ell(u) - 1 + \ell_T(u^{-1}w) + 1 \\
&= \ell(u) + \ell_T(u^{-1}w) \\
&= \ell(w).
\end{aligned}$$

The last equality holds because $u \in C$. Therefore all the inequalities must be equalities, and in particular $\ell(w) = \ell(us) + \ell_T((us)^{-1}w)$. This means that also $us \in C$, which is a contradiction by the minimality of u. So $\ell(us) = \ell(u) + 1$. The same chain of inequalities holds if we change s with t, so we also have $\ell(ut) = \ell(u) + 1$.

Notice now that $wt \in C$, because $\ell(wt) + \ell_T(t) = \ell(w) - 1 + 1 = \ell(w)$. Then, by minimality of u, we have that $\ell(u) \leq \ell(wt) = \ell(w) - 1$. Therefore we can apply the induction hypothesis on the pairs (u, s) and (u, t), and deduce that $u(e_s) > 0$ and $u(e_t) > 0$.

Observe now that $\ell_T(u_T s) \geq \ell_T(u_T)$. Indeed, if this wasn't true then we would have

$$\begin{aligned}
\ell(w) &< \ell(ws) \\
&\leq \ell(u) + \ell(u^{-1}ws) \\
&= \ell(u) + \ell(u_T s) \\
&\leq \ell(u) + \ell_T(u_T s) \\
&< \ell(u) + \ell_T(u_T) \\
&= \ell(w).
\end{aligned}$$

Therefore any reduced expression for u_T in W_T must end with t. A direct computation (which will be omitted here) shows that u_T sends e_s to a non-negative linear combination of $u(e_s)$ and $u(e_t)$. Since $w = uu_T$, we finally have that $w(e_s) > 0$.

To conclude the proof, notice that second part of the thesis follows from the first one applied to ws instead of w. □

In the rest of this section we will examine some interesting consequences of Theorem 1.12, some of which have been anticipated in the previous sections.

Corollary 1.13. Every root is either positive or negative, i.e. $\Phi = \Phi^+ \cup \Phi^-$.

Proof. If $\zeta = w(e_s)$ is a generic root, then we fall in either the first case or in the second case of Theorem 1.12. So we have $\zeta > 0$ or $\zeta < 0$. □

Corollary 1.14. The canonical representation $\rho\colon W \to \mathrm{GL}(V)$ is faithful.

Proof. Let $w \in \ker(\rho)$. Then $w(e_s) = e_s$ for all $s \in S$. Now suppose $w \neq 1$, and choose s so that $\ell(ws) < \ell(w)$. Then by Theorem 1.12 we have $w(e_s) < 0$, which is a contradiction. □

We now turn to the proof of Lemmas 1.4 and 1.9, stated in Section 1.1.

Proof of Lemma 1.4. Let W'_T be the Coxeter group associated to the Coxeter graph $\Gamma|_T$. The canonical representation ρ' of W'_T can be identified with the action of the group generated by $\{\rho_s \mid s \in T\}$, i.e. $\rho(W_T)$, on the subspace V_T of V generated by $\{e_s \mid s \in T\}$. Moreover W'_T can be projected to W_T sending each $s \in T \subseteq W'_T$ to $s \in W_T$. Then the composition

$$W'_T \to W_T \xrightarrow{\rho|_{W_T}} \mathrm{GL}(V_T)$$

coincides with $\rho'\colon W'_T \to \mathrm{GL}(V_T)$. Since ρ' is injective (by Theorem 1.12), the projection $W'_T \to W_T$ is a group isomorphism. □

Proof of Lemma 1.9. We prove by induction on $\ell(w)$ that for every $w \in W_T$ the equality $\ell(w) = \ell_T(w)$ holds. This is obvious if $w = 1$, so let's assume $w \neq 1$. Choose $s \in T$ such that $\ell_T(ws) < \ell_T(w)$. Set $w' = ws$. By Lemma 1.4, the parabolic subgroup W_T is itself a Coxeter group. Moreover its canonical representation is given by the action on V_T obtained restricting $\rho\colon W \to \mathrm{GL}(V)$ to W_T. By Theorem 1.12 applied to W_T we have that $w'(e_s) > 0$. Applying Theorem 1.12 on W we deduce that $\ell(w's) > \ell(w')$, i.e. that $\ell(w) = \ell(w') + 1$. Then, by induction, $\ell(w) = \ell(w') + 1 = \ell_T(w') + 1 = \ell_T(w)$. □

1.4 Positive roots and longest element

In this section we will further investigate the action of a Coxeter group on its roots, especially on the positive ones. We will then derive some important combinatorial consequences about the structure of finite Coxeter groups.

Let $\Pi = \Phi^+$ be the set of positive roots of a fixed Coxeter system (W, S).

Lemma 1.15 ([Hum92], Proposition 5.6)**.** Any $s \in S$ permutes the roots in $\Pi \setminus \{e_s\}$.

Proof. Fix some root $\zeta \in \Pi \setminus \{e_s\}$. Write

$$\zeta = \sum_{r \in S} c_r e_r,$$

for some coefficients $c_r \geq 0$. Since ζ is a unit vector (with respect to the bilinear form B) and $\zeta \neq e_s$, there must be some $t \in S \setminus \{s\}$ such that $c_t > 0$. By definition of the canonical representation, $s(\zeta)$ differs from ζ only by a multiple of e_s. Thus we have

$$s(\zeta) = \sum_{r \in S} c'_r e_r$$

with $c'_r = c_r$ for all $r \neq s$. In particular $c'_t = c_t > 0$, so $s(\zeta)$ is still a positive root different from e_s. Since $s^2 = 1$, we deduce that s permutes the set $\Pi \setminus \{e_s\}$. $\square$

Proposition 1.16 ([Hum92], Proposition 5.6)**.** For any $w \in W$, $\ell(w)$ equals the number of positive roots sent by w to negative roots.

Proof. Call $n(w)$ the number of positive roots sent by w to negative roots, i.e.

$$n(w) = |\Pi(w)|$$

where $\Pi(w) = \Pi \cap w^{-1}(-\Pi)$. Fix any $s \in S$. If $w(e_s) > 0$, then by Lemma 1.15 we have that

$$\begin{aligned}\Pi(ws) &= \Pi \cap sw^{-1}(-\Pi) \\ &= s\big(s(\Pi) \cap w^{-1}(-\Pi)\big) \\ &= s\big((\Pi \setminus \{e_s\} \cup \{-e_s\}) \cap w^{-1}(-\Pi)\big) \\ &= s\big(\big(\Pi \cap w^{-1}(-\Pi)\big) \setminus \big(\{e_s\} \cap w^{-1}(-\Pi)\big) \cup \big(\{-e_s\} \cap w^{-1}(-\Pi)\big)\big) \\ &= s\big(\Pi(w) \setminus \varnothing \cup \{-e_s\}\big) \\ &= s(\Pi(w)) \cup \{e_s\},\end{aligned}$$

so $n(ws) = n(w)+1$. Similarly, if $w(e_s) < 0$ we find that $\Pi(ws) = s(\Pi(w) \setminus \{e_s\})$ and thus $n(ws) = n(w) - 1$. Notice also that $n(1) = \ell(1) = 0$. Then, using Theorem 1.12, we conclude by induction on $\ell(w)$ that $n(w) = \ell(w)$. $\square$

Corollary 1.17 (cf. [Hum92], Theorem 1.8)**.** The action of W on the set

$$\Xi = \{w(\Pi) \mid w \in W\}$$

is simply transitive (i.e. transitive and free).

Proof. It is enough to prove that if $w(\Pi) = \Pi$ then $w = 1$, and this is an immediate consequence of Proposition 1.16. $\square$

Consider now the case where W is finite (e.g. a dihedral group or a symmetric group), so that there is only a finite number of roots. An interesting consequence of the previous results is the existence of exactly one element of maximum length in W. This longest element δ has many interesting properties, and will become important in the study of Artin groups.

Lemma 1.18 (cf. [Hum92], Theorem 1.4)**.** Let W be a finite Coxeter group. Then there exists some element $w \in W$ such that $w(\Pi) = -\Pi$.

Proof. Let $w \in W$ be an element that maximizes the size of $w(\Pi) \cap -\Pi$, or equivalently that minimizes the size of $w(\Pi) \cap \Pi$. Suppose by contradiction that $w(\Pi) \neq -\Pi$, i.e. that $w(\Pi) \cap \Pi \neq \varnothing$. Then the set $\Delta = \{w(e_s) \mid s \in S\}$ cannot be fully contained in $-\Pi$, since otherwise any element of Δ would be a negative combination of the standard basis $\{e_s \mid s \in S\}$ and thus the same would be true for any element of $w(\Pi) \supseteq \Delta$. So there exists some $s \in S$ such that $w(e_s) \in \Pi$, i.e. $e_s \in w(\Pi) \cap \Pi$. By Lemma 1.15, s sends e_s to its negative and permutes all the other positive roots, so $|sw(\Pi) \cap \Pi| = |w(\Pi) \cap \Pi| - 1$. This is a contradiction by definition of w. $\square$

Theorem 1.19 (cf. [Hum92], Theorem 1.8)**.** Let W be a finite Coxeter group. Then there exists a unique element $\delta \in W$ of maximum length. Moreover, the following properties hold:

1. $\delta(\Pi) = -\Pi$;
2. $\ell(\delta) = |\Pi|$;
3. δ has order 2;
4. $\ell(\delta w) = \ell(\delta) - \ell(w)$ for all $w \in W$.

Proof. By Lemma 1.18 and Proposition 1.16 the maximum length of elements of W is exactly $|\Pi|$, and it is realized by all the elements δ such that $\delta(\Pi) = -\Pi$. Furthermore, by Corollary 1.17 there can be only one element δ with this property.

The first two properties are already proved. Since $\delta^2(\Pi) = \delta(-\Pi) = \Pi$, by Corollary 1.17 we have that $\delta^2 = 1$, which is the third property. The roots sent by δw to negative roots are precisely the roots sent by w to positive roots (because δ exchanges positive and negative roots). Therefore, by Proposition 1.16,

$$\ell(\delta w) = |\Pi| - \ell(w) = \ell(\delta) - \ell(w).$$

This concludes the proof of the fourth property. $\square$

1.5 Exchange and Deletion Conditions

In this section we are going to prove two interesting results about the combinatorics of reduced expressions in a Coxeter group. They are called *Exchange Condition* and *Deletion Condition*, respectively.

Theorem 1.20 (cf. [Hum92], Theorem 5.8)**.** Let $w = s_1 \cdots s_r$ for some $w \in W$ and $s_i \in S$. Suppose that $\ell(ws) < \ell(w)$ for some $s \in S$. Then there is an index i such that $ws = s_1 \cdots \hat{s}_i \cdots s_r$.[1] Moreover, if $\ell(w) = r$ (i.e. the expression for w is reduced) then i is unique.

Proof. By Theorem 1.12 we have that $w(e_s) < 0$. Since $e_s > 0$, there exists an index i such that $s_{i+1} \cdots s_r(e_s) > 0$ and $s_i s_{i+1} \cdots s_r(e_s) < 0$. By Lemma 1.15 the only positive root sent by s_i to a negative root is e_{s_i}, thus $s_{i+1} \cdots s_r(e_s) = e_{s_i}$. Therefore the reflection in V about the vector e_{s_i} (with respect to the bilinear form B) is conjugate to the reflection about the vector e_s through the transformation $s_{i+1} \cdots s_r$. In other words:

$$s_i = (s_{i+1} \cdots s_r) s (s_{i+1} \cdots s_r)^{-1}.$$

[1] With the hat notation $\hat{s}_i$, we mean that s_i is omitted.

This implies that

$$\begin{aligned} ws &= s_1 \cdots s_r s \\ &= (s_1 \cdots s_i)(s_{i+1} \cdots s_r)s \\ &= (s_1 \cdots s_i)s_i(s_{i+1} \cdots s_r) \\ &= s_1 \cdots \hat{s}_i \cdots s_r. \end{aligned}$$

Consider now the case $\ell(w) = r$. Suppose by contradiction that there exist indices $i < j$ such that

$$ws = s_1 \cdots \hat{s}_i \cdots s_r = s_1 \cdots \hat{s}_j \cdots s_r.$$

Simplifying $s_1 \cdots s_{i-1}$ on the left and $s_{j+1} \cdots s_r$ on the right we obtain the relation $s_{i+1} \cdots s_j = s_i \cdots s_{j-1}$, which implies that

$$\begin{aligned} w &= s_1 \cdots s_r \\ &= (s_1 \cdots s_i)(s_{i+1} \cdots s_j)(s_{j+1} \cdots s_r) \\ &= s_1 \cdots s_i(s_i \cdots s_{j-1})s_{j+1} \cdots s_r \\ &= s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r. \end{aligned}$$

This is a contradiction since $\ell(w) = r$. □

Theorem 1.20 can be actually generalized (with a similar proof) to the case where s is conjugate to an element of S (see [Hum92]). The resulting proposition is called *Strong Exchange Condition*.

Remark 1.21. The Exchange Condition holds also replacing ws with sw in the statement. Indeed, this latter version can be obtained applying the former to w^{-1}.

Theorem 1.22 ([Hum92], Corollary 5.8)**.** Let $w = s_1 \cdots s_r$ for some $w \in W$ and $s_i \in S$, with $\ell(w) < r$. Then there exist indices $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$.

Proof. Since $\ell(w) < r$, there exists an index j such that $\ell(w's_j) < \ell(w')$, where $w' = s_1 \cdots s_{j-1}$. By the Exchange Condition (Theorem 1.20) applied to w' and s_j, there exists an index $i < j$ such that $w's_j = s_1 \cdots \hat{s}_i \cdots s_{j-1}$. Multiplying both sides of this equality by $s_{j+1} \cdots s_r$ on the right, we obtain the desired result. □

The Deletion Condition we have just stated has the following immediate consequence.

Corollary 1.23 ([Hum92], Corollary 5.8)**.** Let $w = s_1 \cdots s_r$ for some $w \in W$ and $s_i \in S$. Then a reduced expression for w can be obtained omitting an even number of s_i in the previous expression.

1.6 Minimal coset representatives

We will now prove a few interesting properties of standard parabolic subgroups which will become useful later. As usual, fix a Coxeter system (W, S). For any $T \subseteq S$ define

$$W^T = \{w \in W \mid \ell(ws) > \ell(w)\ \forall\, s \in T\}.$$

By the results of Section 1.5, the set W^T consists of all the elements of W admitting no reduced expression ending with elements of T.

Proposition 1.24 ([Hum92], Proposition 1.10)**.** Let $T \subseteq S$. Then for any $w \in W$ there exist a unique $u \in W^T$ and a unique $v \in W_T$ such that $w = uv$. Moreover, $\ell(w) = \ell(u) + \ell(v)$ and u is the unique element of smallest length in the coset wW_T.

Proof. Let u be an element of smallest length in the coset wW_T, and let $v = wu^{-1} \in W_T$. For any $s \in T$, we have that $us \in wW_T$ and thus $\ell(us) > \ell(u)$. This means that $u \in W^T$. Consider now reduced expressions

$$u = s_1 \cdots s_q, \quad v = s'_1 \cdots s'_r$$

with $q = \ell(u)$, $r = \ell(v)$, $s_i \in S$ and $s'_i \in T$ (the last condition can be fulfilled by Lemma 1.9). Suppose by contradiction that $\ell(w) < \ell(u) + \ell(r) = q + r$. Then, by the Deletion Condition (Theorem 1.22) it is possible to omit two factors in the expression $s_1 \cdots s_q s'_1 \cdots s'_r$ without changing w. Omitting any factor s_i would give rise to an element of wW_T of length smaller than $q = \ell(u)$, which is impossible by definition of u. On the other hand, if it were possible to omit two factors s'_i, s'_j we would obtain an expression for v of length smaller than $r = \ell(v)$, which is also impossible. Therefore $\ell(w) = \ell(u) + \ell(r)$.

With the same argument it follows that any element $w' \in wW_T$ can be written in the form uv' for some $v' \in W_T$ (but with the same u as before) and the property $\ell(w') = \ell(u) + \ell(v')$ is satisfied. In particular when $v' \neq 1$ we obtain that $\ell(w') > \ell(u)$, which means that u is the unique element of smallest length in wW_T.

Finally we have to prove the uniqueness of u in $W^T \cap wW_T$. If there was some other element $u' \in W^T \cap wW_T$, then we could write $u' = uv$ for some $v \in W_T \setminus \{1\}$ with $\ell(u') = \ell(u) + \ell(v)$. But choosing $s \in T$ such that $\ell(vs) < \ell(v)$ we would obtain $\ell(u's) < \ell(u')$, so $u' \notin W^T$. $\square$

The following corollary is part of the statement of the previous proposition, but we want to underline it as an interesting result by itself.

Corollary 1.25. Let $T \subseteq S$. For any $w \in W$ there exists a unique element of smallest length in the coset wW_T.

1.7 Dual representation and Tits cone

In this section we are going to investigate some geometric properties of the dual of the canonical representation $\rho\colon W \to \mathrm{GL}(V)$, i.e. the representation

$$\rho^*\colon W \to \mathrm{GL}(V^*)$$

given by $\rho^*(w) = (\rho(w)^t)^{-1}$. The notation $\rho(w)(f)$ for $w \in W$ and $f \in V^*$ will be shortened into $w(f)$ throughout this section, similarly to what we already do for the canonical representation. If $f \in V^*$ and $v \in V$, denote $f(v)$ by $\langle f, v\rangle$. By definition, the dual representation ρ^* satisfies

$$\langle w(f), w(v)\rangle = \langle f, v\rangle.$$

The following is an immediate consequence of Corollary 1.14.

Proposition 1.26. The dual representation $\rho^*\colon W \to \mathrm{GL}(V^*)$ is faithful.

For $s \in S$ consider the hyperplane

$$H_s = \{f \in V^* \mid \langle f, e_s\rangle = 0\}$$

of V^*, together with the two half-spaces A_s^+ and A_s^- defined by

$$A_s^+ = \{f \in V^* \mid \langle f, e_s\rangle > 0\}, \quad A_s^- = \{f \in V^* \mid \langle f, e_s\rangle < 0\}.$$

Any $f \in V^*$ is uniquely determined by its value on e_s and on all the vectors in the hyperplane $L_s \subseteq V$ fixed by $\rho(s)$. Then, if $f \in H_s$, we have that

$$\begin{aligned} \langle s(f), e_s\rangle &= \langle s(f), -s(e_s)\rangle = -\langle f, e_s\rangle = 0; \\ \langle s(f), v\rangle &= \langle s(f), s(v)\rangle = \langle f, v\rangle \quad \forall\, v \in L_s. \end{aligned}$$

This means that s fixes H_s pointwise. Moreover, since $\langle s(f), e_s\rangle = -\langle f, e_s\rangle$, it sends A_s^+ to A_s^- and vice versa. Indeed s acts on V^* as a linear reflection, fixing the hyperplane H_s and sending f to $-f$ for all $f \in V^*$ such that $f|_{L_s} = 0$.

Call C_0 the intersection of the half-spaces A_s^+ for $s \in S$. Since the A_s^+ are open (with respect to the only topology that makes V^* a topological vector space) the intersection C_0 is also open.

Lemma 1.27 ([Hum92], Lemma 5.13)**.** Let $s \in S$ and $w \in W$. Then $\ell(sw) > \ell(w)$ if and only if $w(C_0) \subseteq A_s^+$, and $\ell(sw) < \ell(w)$ if and only if $w(C_0) \subseteq A_s^-$.

Proof. We have that $\ell(sw) > \ell(w)$ is equivalent to $\ell(w^{-1}s) > \ell(w^{-1})$, which is equivalent to $w^{-1}(e_s) > 0$ by Theorem 1.12. Let $f \in C$. Then $\langle w(f), e_s\rangle > 0$ is equivalent to $\langle f, w^{-1}(e_s)\rangle > 0$, which is equivalent (by definition of C) to $w^{-1}(e_s) > 0$. So we conclude that the relation $\ell(sw) > \ell(w)$ holds if and only if $\langle w(f), e_s\rangle > 0$ i.e. $w(f) \in A_s^+$. The second part of the statement easily follows. $\square$

Proposition 1.28 (cf. [Hum92], Theorem 5.13)**.** For all $w \in W \setminus \{1\}$ we have that $w(C_0) \cap C_0 = \varnothing$.

Proof. If $w \neq 1$ then there exists some $s \in S$ such that $\ell(sw) < \ell(w)$. Then, by Lemma 1.27, $w(C_0) \subseteq A_s^-$. Since $C_0 \subseteq A_s^+$ and $A_s^+ \cap A_s^- = \varnothing$, we deduce that $w(C_0) \cap C_0 = \varnothing$. □

Corollary 1.29. The action of W on the set $\{w(C_0) \mid w \in W\}$ is free and transitive.

Consider now the subset of V^* given by

$$I = \bigcup_{w \in W} w(\bar{C}_0).$$

Since C_0 is a cone, I is also a cone. It is called *Tits cone* of the Coxeter system (W, S).

Proposition 1.30. The Tits cone I is convex.

Proof. Let $f, g \in I$. We want to prove that the closed segment $[f, g]$ joining f and g in V^* is contained in I. We can assume without loss of generality that $f \in \bar{C}_0$ and $g \in w(\bar{C}_0)$ for some $w \in W$ (otherwise, if $f \in w'(\bar{C}_0)$, we can apply w'^{-1} to both f and g in order to have $w'^{-1}(f) \in \bar{C}_0$). We are going to prove the thesis by induction on $\ell(w)$.

If $w = 1$ there is nothing to prove, since $\bar{C}_0$ is convex. Assume from now on that $w \neq 1$. Then the segment $[f, g]$ intersects $\bar{C}_0$ in some segment $[f, h]$, for some $h \in \partial C_0$. Let $T = \{s \in S \mid g \in A_s^-\}$. If we had $h \in A_s^+$ for all $s \in T$, then all points k in an open neighbourhood of h in $[f, g]$ would satisfy both $k \in A_s^+$ (because A_s^+ is open) and $k \in \overline{A_s^+}$ (because the closed half-space $\overline{A_s^+}$ is convex and contains both f and g, so it contains the entire segment $[f, g]$). Therefore $h \in H_s$ for some $s \in T$. Notice that $g \in A_s^-$, so $w(C_0) \subseteq A_s^-$. By Lemma 1.27 this means that $\ell(sw) < \ell(w)$. Apply s to both h and g. Then $h \in \bar{C}_0$ and $g \in sw(\bar{C}_0)$, thus by induction hypothesis the entire segment $[h, g]$ is contained in I. □

In general the Tits cone is strictly contained in V^*. More precisely, it can be seen that $I = V^*$ if and only if W is finite (see [Hum92], Section 5.13). The Tits cone will be used in Chapter 4 to formulate the $K(\pi, 1)$ conjecture.

1.8 Classification of finite Coxeter groups

Definition 1.31. A Coxeter system (W, S) is said to be of *finite type* if the Coxeter group W is finite. In this case we also say that the corresponding Coxeter graph is of finite type.

As we will see, Coxeter systems of finite type play a prominent role in the theory of Coxeter and Artin groups. Their classification was first derived by Coxeter [Cox34]. Although in the following chapters we will not strictly need such classification, we are going to present it (without proofs) in order to have a better insight into the theory of Coxeter groups.

Notice that, since a Coxeter group is the direct product of its maximal irreducible parabolic subgroups, it is enough to classify the finite irreducible Coxeter groups. Then any finite Coxeter group will be obtained as a direct product of irreducible components.

Theorem 1.32 ([Hum92], Corollary 6.2 and Theorem 6.4)**.** A Coxeter system (W, S) is of finite type if and only if the bilinear form $B\colon V \times V \to \mathbb{R}$ of Section 1.2 is positive definite.

Theorem 1.33 ([Hum92], Theorem 2.7 and Theorem 6.4)**.** The irreducible Coxeter graphs of finite type are precisely those listed in Figure 1.3.

Remark 1.34. Most of the Coxeter graphs of Figure 1.3 correspond to the so called *Dynkin diagrams*, which arise in other branches of mathematics such as Lie theory (in the classification semisimple Lie algebras).

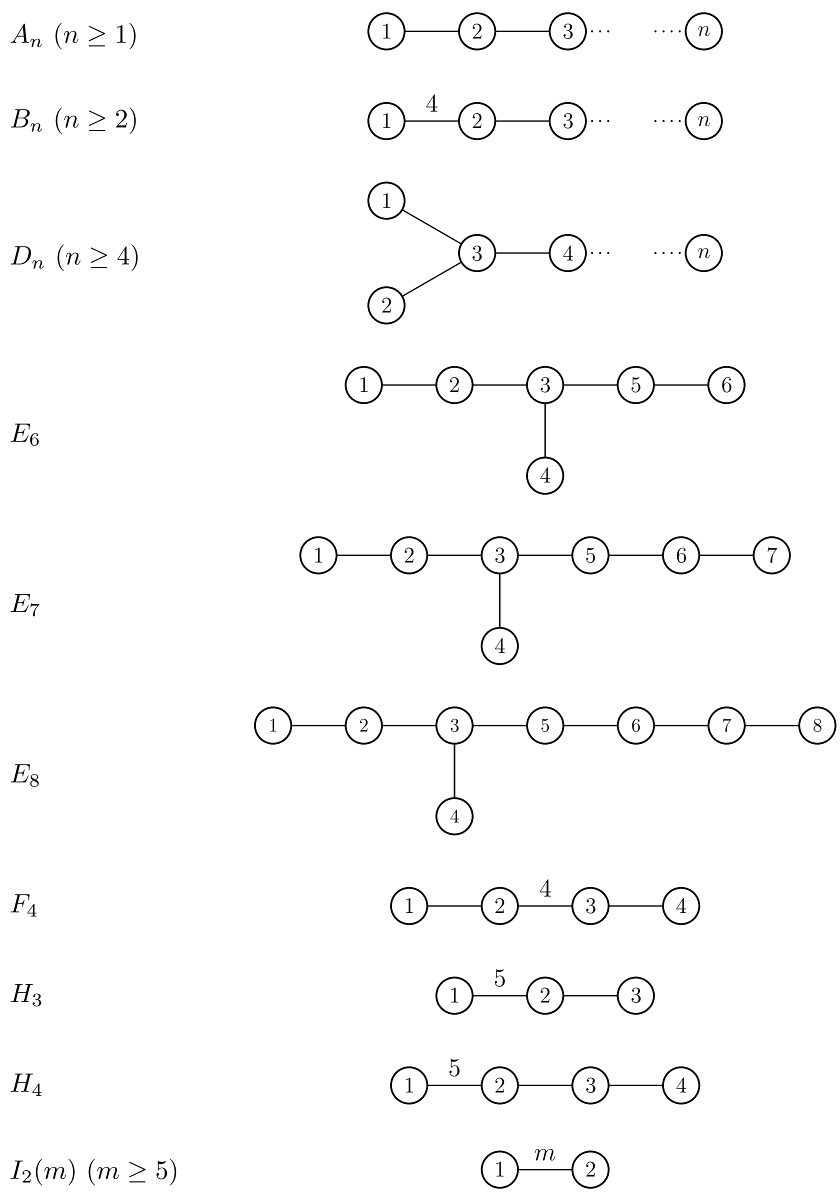


Figure 1.3: Irreducible Coxeter graphs of finite type.

Chapter 2

Artin groups

Artin groups were introduced in their full generality by Tits [Tit66], and the first deep study of their properties was made by Brieskorn and Saito [BS72]. Further research has been carried out in more recent years, but still there isn't a wide and classical general theory of Artin groups as there is for Coxeter groups.

In this chapter we are going to define Artin groups and we are going to present some of their known properties.

2.1 Definition and relation with Coxeter groups

Consider a Coxeter graph (Γ, S). Recall that the corresponding Coxeter group is defined as

$$W_\Gamma = \langle\, S \mid (st)^{m_{s,t}} = 1 \;\; \forall\, s,t \in S \text{ such that } m_{s,t} \neq \infty \,\rangle.$$

Since in W_Γ all the generators of S have order 2, the relations $(st)^{m_{s,t}} = 1$ for $s \neq t$ can be also written as

$$\Pi(s,t,m_{s,t}) = \Pi(t,s,m_{s,t}),$$

where the notation $\Pi(a,b,m)$ stands for

$$\Pi(a,b,m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even,} \\ (ab)^{\frac{m-1}{2}}a & \text{if } m \text{ is odd.} \end{cases}$$

For instance, if $m_{s,t} = 3$ the relation $(st)^3 = 1$ can be written as $sts = tst$. So we have that

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{ll} s^2 = 1 & \forall\, s \in S, \\ \Pi(s,t,m_{s,t}) = \Pi(t,s,m_{s,t}) & \forall\; s,t \in S \text{ such that } m_{s,t} \neq \infty \end{array} \right\rangle.$$

Consider now the set $\Sigma = \{\sigma_s \mid s \in S\}$, which is in natural bijection with S. We will use this set as a generating set for the Artin group A_Γ, as follows.

Definition 2.1. The Artin group A_Γ corresponding to the Coxeter graph (Γ, S) is the group presented as

$$A_\Gamma = \langle\, \Sigma \mid \Pi(\sigma_s, \sigma_t, m_{s,t}) = \Pi(\sigma_t, \sigma_s, m_{s,t}) \;\forall\; s,t \in S \text{ such that } m_{s,t} \neq \infty \rangle.$$

We also call the pair (A_Γ, Σ) an *Artin system.* As for Coxeter graphs, we say that an Artin group A_Γ is of finite type if Γ is of finite type.

For example, if Γ is of type A_{n-1}, the corresponding Artin group is called *braid group on n strands* and is denoted by $\mathfrak{B}_n$ (see [KDT08]). Braid groups were first defined on their own in the 1920s, and motivated the general definition of Artin groups. They have strong connections with knot theory, and can be obtained as fundamental groups of the configuration spaces of n undistinguished points in the plane:

$$\mathfrak{B}_n = \pi_1 \left(\left(\mathbb{C}^n \setminus \bigcup_{i \neq j} \{x_i = x_j\} \right) / \mathfrak{S}_n \right),$$

where $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ permuting the coordinates. As we will see in Chapter 4, all Artin groups admit a similar interpretation.

For any Coxeter graph Γ there is a natural projection $\pi\colon A_\Gamma \to W_\Gamma$, sending σ_s to s for all $s \in S$. The kernel of this projection is often called *colored Artin group*, and is denoted by CA_Γ. This leads to the following short exact sequence of groups:

$$1 \to CA_\Gamma \to A_\Gamma \xrightarrow{\pi} W_\Gamma \to 1.$$

We are now going to construct a natural set-section $\tau\colon W_\Gamma \to A_\Gamma$, i.e. a function such that $\pi \circ \tau = \mathrm{id}_{W_\Gamma}$. This will not be a group homomorphism (except in the trivial case $S = \varnothing$). Indeed, since Artin groups are conjectured to be torsion-free (see Section 4.2), there shouldn't exist non-trivial homomorphisms $W_\Gamma \to A_\Gamma$ at all. In order to construct τ we will need the following result about reduced expressions in Coxeter groups.

Definition 2.2. Let $w \in W_\Gamma$, and let $\mu, \mu' \in S^*$ be two expressions for w. We say that there is an *elementary transformation* joining μ and μ' if there exist $\nu_1, \nu_2 \in S^*$ and $s, t \in S$ such that $m_{s,t} \neq \infty$,

$$\mu = \nu_1\, \Pi(s,t,m_{s,t})\, \nu_2 \quad \text{and} \quad \mu' = \nu_1\, \Pi(t,s,m_{s,t})\, \nu_2.$$

Theorem 2.3 ([Tit69]; cf. [Bro89])**.** Let $w \in W_\Gamma$, and let $\mu, \mu' \in S^*$ be two reduced expressions for w. Then there exists a finite sequence of elementary transformations joining μ and μ'.

Proof. The proof is by induction on $k = \ell(w)$, the case $k = 0$ being trivial since there is only one expression of length 0. Assume then $k > 0$, and let $\mu = s_1 \cdots s_k$, $\mu' = t_1 \cdots t_k$. Set $s = s_1$ and $t = t_1$. If $s = t$ we are done by applying the induction hypothesis to the reduced expressions $s_2 \cdots s_k$ and $t_2 \cdots t_k$. Therefore, suppose from now on that $s \neq t$.

Our aim is to prove that w admits a reduced expression starting with $\Pi(s, t, m_{s,t})$. Let h be the the maximum nonnegative integer such that w admits a reduced expression ν starting with $\Pi(s, t, h)$ or with $\Pi(t, s, h)$. If $h \geq m_{s,t}$ we are done. Assume then by contradiction that $h < m_{s,t}$, and suppose without loss of generality that ν starts with $\Pi(s, t, h)$:

$$\nu = \Pi(s, t, h)\, r_{h+1} \cdots r_k \quad (\text{for some } r_i \in S).$$

Since w admits a reduced expression starting with t (namely, μ'), by the Exchange Condition (Theorem 1.20) applied to ν there exists a reduced expression for w obtained from ν by adding a t at the beginning and omitting some other letter of ν. There are two cases.

- The omitted letter lies in the prefix $\Pi(t, s, h)$. It cannot be the initial letter, because $s \neq t$. It cannot be the final letter, because otherwise we would obtain that

  $$s\, \Pi(t, s, h-1)\, r_{h+1} \cdots r_k = \Pi(t, s, h)\, r_{h+1} \cdots r_k,$$

  i.e. $\Pi(s, t, h) = \Pi(t, s, h)$, which is impossible by Lemma 1.2 since $h < m_{s,t}$. Finally the omitted letter cannot lie in the interior of $\Pi(t, s, h)$, because its omission would leave two consecutive s or two consecutive t, and the resulting expression would not be reduced. In any case, we obtain a contradiction.

- The omitted letter lies in the suffix $r_{h+1} \cdots r_k$. Then we get

  $$w = s\, \Pi(t, s, h)\, r_{h+1} \cdots \hat{r}_i \cdots r_k = \Pi(s, t, h+1)\, r_{h+1} \cdots \hat{r}_i \cdots r_k.$$

  So there is a reduced expression for w starting with $\Pi(s, t, h+1)$, which is a contradiction by maximality of h.

So we have proved that w admits a reduced expression ν starting with $\Pi(s, t, m_{s,t})$, and in particular $m_{s,t} \neq \infty$. Replacing the prefix $\Pi(s, t, m_{s,t})$ with $\Pi(t, s, m_{s,t})$ in ν, we obtain a reduced expression ν' for w starting with $\Pi(t, s, m_{s,t})$. Then we can construct a sequence of elementary transformations joining μ and μ' as follows: first we transform μ into ν (they both start with an s, so the induction hypothesis applies); then we transform ν into ν' (they differ by a single elementary transformation); finally we transform ν' into μ' (again, applying the induction hypothesis thanks to the fact that ν' and μ' both start with a t). $\square$

We are now ready to define the set-section $\tau\colon W_\Gamma \to A_\Gamma$. Given an element $w \in W_\Gamma$, consider a reduced expression $s_1 \cdots s_k$ for w and set

$$\tau(w) = \sigma_{s_1} \cdots \sigma_{s_k}.$$

By Theorem 2.3 different reduced expressions for w yield the same element of A_Γ, so τ is well-defined. Moreover it is a right-inverse of $\pi\colon A_\Gamma \to W_\Gamma$, since

$$\pi(\tau(w)) = \pi(\sigma_{s_1} \cdots \sigma_{s_k}) = \pi(\sigma_{s_1}) \cdots \pi(\sigma_{s_k}) = s_1 \cdots s_k = w.$$

2.2 Artin monoids

Definition 2.4. The *Artin monoid* corresponding to a Coxeter graph (Γ, S) is the monoid presented as

$$A_\Gamma^+ = \langle\, \Sigma \mid \Pi(\sigma_s, \sigma_t, m_{s,t}) = \Pi(\sigma_t, \sigma_s, m_{s,t}) \;\forall\; s,t \in S \text{ such that } m_{s,t} \neq \infty \rangle.$$

Notice that this presentation is identical to that of the Artin group A_Γ, but A_Γ^+ needs not contain inverses of its elements (as we will see, only the identity $1 \in A_\Gamma^+$ has an inverse). Artin monoids are extremely important in the study of the corresponding Artin groups. Most of the literature on this topic was developed in [BS72, Gar69, Par02]. In this section we present, without proofs, the most important results about Artin monoids and about their relationship with Artin groups.

The reason why we take the freedom to use the same generating set Σ for A_Γ^+ and for A_Γ is given by the following theorem.

Theorem 2.5 ([Par02])**.** The natural monoid homomorphism $A_\Gamma^+ \to A_\Gamma$ is injective.

In view of Theorem 2.5, from now on we will consider A_Γ^+ as contained in A_Γ. The Artin monoid is also called *positive monoid* of A_Γ, for its elements are precisely those which can be written as a product (with positive exponents) of generators in Σ. An immediate consequence of the previous theorem is the following.

Corollary 2.6 ([BS72])**.** The Artin monoid A_Γ^+ is cancellative, i.e. $\alpha\gamma_1\beta = \alpha\gamma_2\beta$ implies $\gamma_1 = \gamma_2$ for all $\alpha, \beta, \gamma_1, \gamma_2 \in A_\Gamma^+$.

Since the relations $\Pi(\sigma_s, \sigma_t, m_{s,t}) = \Pi(\sigma_t, \sigma_s, m_{s,t})$ involve the same number of generators on the left hand side and on the right hand side, there is a well defined length function $\ell\colon A_\Gamma^+ \to \mathbb{N}$ that sends an element $\sigma_{s_1} \cdots \sigma_{s_k} \in A_\Gamma^+$ to the length k of its representation. Clearly we have that

$$\ell(\alpha\beta) = \ell(\alpha) + \ell(\beta) \quad \forall\, \alpha, \beta \in A_\Gamma^+,$$

i.e. ℓ is a monoid homomorphism. An easy consequence is that the identity $1 \in A_\Gamma^+$ is the only invertible element of the Artin monoid.

Notice that the restriction of $\pi\colon A_\Gamma \to W_\Gamma$ to the positive monoid A_Γ^+ is still surjective, and that the set-section τ defined in Section 2.1 has image contained in A_Γ^+. Moreover τ is a length-preserving function: $\ell(\tau(w)) = \ell(w)$ for all $w \in W_\Gamma$.

Definition 2.7. Given $\alpha, \beta \in A_\Gamma^+$, we say that $\alpha \preceq_{\mathrm{L}} \beta$ if there exists $\gamma \in A_\Gamma^+$ such that $\alpha\gamma = \beta$. Similarly we say that $\alpha \preceq_{\mathrm{R}} \beta$ if there exists $\gamma \in A_\Gamma^+$ such that $\gamma\alpha = \beta$.

If $\alpha \preceq_{\mathrm{L}} \beta$ we also say that α is a left divisor of β, that α left divides β, or that β is left divisible by α. We do the same for right divisibility.

Lemma 2.8. Both $\preceq_{\mathrm{L}}$ and $\preceq_{\mathrm{R}}$ are partial order relations on A_Γ^+.

Proof. Reflexivity and transitivity are obvious. For antisymmetry, assume $\alpha \preceq_{\mathrm{L}} \beta$ and $\beta \preceq_{\mathrm{L}} \alpha$. Then $\alpha\gamma_1 = \beta$ and $\beta\gamma_2 = \alpha$ for some $\gamma_1, \gamma_2 \in A_\Gamma^+$, thus $\alpha = \alpha\gamma_1\gamma_2$. By cancellativity (Corollary 2.6) this implies that $\gamma_1\gamma_2 = 1$, therefore $\gamma_1 = \gamma_2 = 1$, i.e. $\alpha = \beta$. The same argument applies to $\preceq_{\mathrm{R}}$. $\square$

Definition 2.9. Let E be a subset of A_Γ^+. A *left common divisor* of E is any element of A_Γ^+ which left divides all elements of E. A *greatest left common divisor* (or *left g.c.d.*) of E is a left common divisor of E which is left multiple of all the left common divisors of E. Similarly, a *left common multiple* of E is any element of A_Γ^+ which is left multiple of all elements of E, and a *left least common multiple* (or *left l.c.m.*) of E is a left common multiple of E which is left divisible for all the left common multiples of E. Define in the obvious way the analogous concepts for right divisibility.

Notice that when a greatest common divisor or a least common multiple exists for a set E, then it is unique. Indeed, suppose for instance that α and β are greatest left common divisors of E; then $\alpha \preceq_{\mathrm{L}} \beta$ and $\beta \preceq_{\mathrm{L}} \alpha$, which implies $\alpha = \beta$ by Lemma 2.8.

Proposition 2.10 ([BS72])**.** Let E be a subset of A_Γ^+. If E admits a left (resp. right) common multiple, then it also admits a least left (resp. right) common multiple.

Proposition 2.11 ([BS72])**.** Any non-empty subset E of A_Γ^+ admits a greatest left common divisor and a greatest right common divisor.

We are now going to introduce the fundamental element of the Artin monoid, which is (when it exists) significantly important. Recall that $\Sigma = \{\sigma_s \mid s \in S\}$.

Theorem 2.12 ([BS72])**.** For an Artin monoid A_Γ^+, the following conditions are equivalent:

- Γ is of finite type;
- Σ admits a least left common multiple;
- Σ admits a least right common multiple.

Moreover, if they are satisfied, then the least left common multiple and the least right common multiple of Σ coincide.

Definition 2.13. If Γ is a Coxeter graph of finite type, the least left (or right) common multiple of Σ in A_Γ^+ is called *fundamental element* of A_Γ^+ and is usually denoted by Δ.

The following theorem summarizes some of the properties of the fundamental element. Before that, two more definitions are required.

Definition 2.14. An element $\alpha \in A_\Gamma^+$ is *square-free* if it cannot be written in the form $\beta\sigma_s^2\gamma$ for $\beta, \gamma \in A_\Gamma^+$ and $s \in S$.

Definition 2.15. Let $\operatorname{rev}\colon A_\Gamma^+ \to A_\Gamma^+$ be the bijection that sends an element $\sigma_{s_1}\sigma_{s_2}\cdots\sigma_{s_k} \in A_\Gamma^+$ to its "reverse" $\sigma_{s_k}\sigma_{s_{k-1}}\cdots\sigma_{s_1}$. It is easy to check that it is well defined.

Theorem 2.16 ([BS72])**.** Let Γ be a Coxeter graph of finite type, so that A_Γ^+ admits a fundamental element Δ. Then the following properties hold:

(i) $\operatorname{rev}\Delta = \Delta$;

(ii) an element of A_Γ^+ is square-free if and only if it is a (left or right) divisor of Δ;

(iii) an element of A_Γ^+ left divides Δ if and only if it right divides Δ;

(iv) the least (left or right) common multiple of square-free elements of A_Γ^+ is square-free;

(v) Δ is the uniquely determined square-free element of maximal length in A_Γ^+;

(vi) $\Delta = \tau(\delta)$, where δ is the longest element of W_Γ;

(vii) any element $\alpha \in A_\Gamma$ can be written in the form $\alpha = \Delta^{-k}\beta$ for some $\beta \in A_\Gamma^+$ and $k \in \mathbb{N}$.

Some of the properties of Theorem 2.16 are easy to justify: (i) follows from the last part of Theorem 2.12, whereas (iii), (iv) and (v) follow from (ii).

Property (vi) of Theorem 2.16 is enough to find Δ in some simple cases. For instance, if all the generators σ_s commute (i.e. $m_{s,t} = 2$ for all $s \neq t$ in S, so A_Γ is free abelian) then Δ is simply the product of the generators. Instead, if S consists of only two elements s and t (i.e. the Coxeter group W_Γ is a dihedral group), then $\Delta = \Pi(\sigma_s, \sigma_t, m_{s,t})$.

2.3 Standard parabolic subgroups and normal form

We are going to define standard parabolic subgroups of an Artin group, similarly to how we defined those of a Coxeter group (cf. Definition 1.3). Let (Γ, S) be a Coxeter graph with Artin system (A_Γ, Σ), and let $T \subseteq S$. Set $A = A_\Gamma$ and $A^+ = A_\Gamma^+$, for simplicity.

Definition 2.17. Let $\Sigma_T = \{\sigma_s \mid s \in T\}$ and let A_T be the subgroup of A generated by Σ_T. A subgroup constructed in this way is called *standard parabolic subgroup* of A.

Theorem 2.18 ([vdL83])**.** The natural homomorphism $A_T \to A$ which sends σ_s to σ_s for all $s \in T$ is injective. In other words, (A_T, Σ_T) is the Artin system corresponding to the Coxeter graph $(\Gamma|_T, T)$.

Theorem 2.19 ([BS72])**.** The least (left or right) common multiple of Σ_T exists in A^+ if and only if the Coxeter graph $(\Gamma|_T, T)$ is of finite type.

If $\Gamma|_T$ is a Coxeter graph of finite type, it makes sense to consider the fundamental element of the Artin monoid A_T^+ corresponding to the Artin system (A_T, Σ_T). Such element will be denoted by Δ_T.

Lemma 2.20 ([BS72])**.** Δ_T is precisely the least (left or right) common multiple of Σ_T in A^+.

In the rest of this section we are going to introduce a normal form for elements of the Artin monoid A^+. To do so, define for any $\alpha \in A^+$ the set

$$I(\alpha) = \{s \in S \mid \alpha = \beta\sigma_s \text{ for some } \beta \in A^+\}.$$

In other words, this is the set of elements $s \in S$ such that σ_s right divides α.

Theorem 2.21 ([BS72])**.** For any $\alpha \in A^+$ there exists a unique tuple $(T_1, \dots, T_k)$ of non-empty subsets of S such that

$$\alpha = \Delta_{T_k}\Delta_{T_{k-1}} \cdots \Delta_{T_1}$$

and $I(\Delta_{T_k} \cdots \Delta_{T_j}) = T_j$ for $1 \leq j \leq k$.

Proof. The proof is by induction on $\ell(\alpha)$, the case $\ell(\alpha) = 0$ being trivial (k must be equal to 0). Assume then $\ell(\alpha) > 0$. If $T_1, \dots, T_k$ are as in the statement, then we must have $k > 0$ and (by the last property for $j = 1$)

$$T_1 = I(\Delta_{T_k} \cdots \Delta_{T_1}) = I(\alpha).$$

So T_1 is uniquely determined. Moreover α is right divisible by all the elements in $T_1 = I(\alpha)$, and thus it is right divisible by their least right common multiple Δ_{T_1}. Set $\alpha = \beta\Delta_{T_1}$. Then existence and uniqueness of $T_2, \cdots, T_k$ follow applying the induction hypothesis on β. $\square$

Chapter 3

Discrete Morse theory

Discrete Morse theory is a powerful tool for simplifying CW complexes while maintaining their homotopy type. It was first developed by Forman [For98], who presented it as a combinatorial analogue of Morse theory. Forman's version of discrete Morse theory, based on the concept of discrete Morse function, was later reformulated by Chari and Batzies in terms of acyclic matchings [BW02, Cha00]. In this chapter we are going to present the latter formulation, with a few examples.

3.1 Face poset and acyclic matchings

Let X be a CW complex. Recall that each cell of X has a characteristic map $\Phi\colon D^n \to X$ and an attaching map $\varphi\colon S^{n-1} \to X$, where $\varphi = \Phi|_{\partial D^n}$ (see [Hat02]).

Definition 3.1. The *face poset* of X is the set $X^{(*)}$ of its cells together with the partial order defined by $\sigma \leq \tau$ if $\bar{\sigma} \subseteq \bar{\tau}$.

Definition 3.2 ([For98])**.** Let $\sigma, \tau \in X^{(*)}$. If $\dim \tau = \dim \sigma + 1$ and $\sigma \leq \tau$ we say that σ is a *face* of τ. We say that σ is a *regular face* of τ if, in addition, the two following conditions hold (set $n = \dim \sigma$ and let Φ be the attaching map of τ):

(i) $\Phi|_{\Phi^{-1}(\sigma)}\colon \Phi^{-1}(\sigma) \to \sigma$ is a homeomorphism;

(ii) $\overline{\Phi^{-1}(\sigma)}$ is homeomorphic to D^n.

Definition 3.3 ([For98])**.** X is a *regular CW complex* if all the attaching maps are injective.

Remark 3.4. If X is regular, then all its faces are regular.

In order to state the main results of discrete Morse theory, we need to introduce matchings on the cell graphs of CW complexes. The required definitions follow.

Definition 3.5. The *cell graph* G_X of X is the Hasse diagram of $X^{(*)}$, i.e. a directed graph with $X^{(*)}$ as set of vertices and an edge from τ to σ (written $\tau \to \sigma$) if σ is a face of τ. Denote the set of edges of G_X by E_X.

Definition 3.6. A *matching* on X is a subset $M \subseteq E_X$ such that

(i) if $(\tau \to \sigma) \in M$, then σ is a regular face of τ;

(ii) any cell of X occurs in at most one edge of M.

Given a matching M on X, define a graph G_X^M obtained from G_X by inverting all the edges in M.

Definition 3.7. A matching M on X is *acyclic* if the corresponding graph G_X^M is acyclic.

For example, consider the torus $X = S^1 \times S^1$ with the structure of CW complex shown in Figure 3.1 on the left. Such complex has one 0-cell x, three 1-cells a, b and c, and two 2-cells A and B. The corresponding cell graph is shown on the right. The regular faces are exactly those dotted on the left of Figure 3.2. A possible matching on X is shown in the middle of Figure 3.2. Such matching is not acyclic, since the corresponding graph G_X^M has the cycle $A \to c \to B \to a \to A$. It is easy to check that any acyclic matching on X has at most one edge. An example of acyclic matching is shown on the right of Figure 3.2.

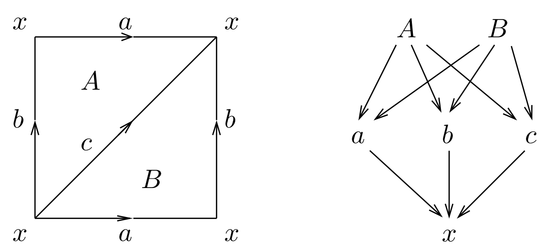


Figure 3.1: A CW structure for the torus (on the left) and the corresponding cell graph (on the right).

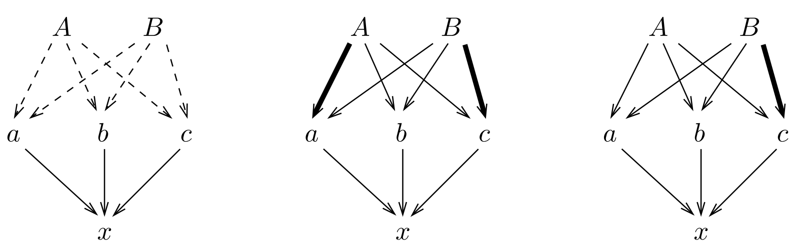


Figure 3.2: Regular edges (dotted on the left), a matching (bold in the middle) and an acyclic matching (bold on the right).

3.2 The Morse complex

The aim of discrete Morse theory is to construct, from a CW complex X with an acyclic matching M, a simpler CW complex X_M (called *Morse complex*) homotopy equivalent to X but with fewer cells. In this section we are going to state the main theorem of discrete Morse theory (the existence of such Morse complex) with a sketch of its proof in the case of finite complexes. Then we are going to present an example which better clarifies the situation in the case of 2-dimensional complexes, and finally we are going to prove a simple lemma which is useful to construct acyclic matchings.

Definition 3.8. Let M be an acyclic matching on X. A cell of X is *M-essential* if it doesn't occur in any edge of M.

For example, the essential cells of the matching on the right of Figure 3.2 are A, a, b and x.

Definition 3.9. Let $(P, \leq)$ be a poset. A *P-grading* on X is a poset map $\eta\colon X^{(*)} \to P$. Given a P-grading on X, for any $p \in P$ denote by $X_{\leq p}$ the subcomplex of X consisting of all the cells σ such that $\eta(\sigma) \leq p$.

Definition 3.10. A P-grading on X is *compact* if $X_{\leq p}$ is compact for all $p \in P$.

Definition 3.11. Let M be an acyclic matching on X and η a P-grading on X. We say that M and η are *compatible* if $\eta(\sigma) = \eta(\tau)$ for all $(\tau \to \sigma) \in M$. In other words, the matching M can be written as union of matchings M_p for $p \in P$, where each M_p is a matching on the fiber $\eta^{-1}(p)$.

Theorem 3.12 ([BW02])**.** *Let X be a CW complex with an acyclic matching M and a compact P-grading η such that M and η are compatible. Then there exist a CW complex X_M, with n-cells in one-to-one correspondence with the M-essential n-cells of X, and a homotopy equivalence $f_M\colon X \to X_M$. Moreover such construction is natural with respect to inclusion: let Y be a subcomplex of X such that, if $(\tau \to \sigma) \in M$ and $\sigma \in Y^{(*)}$, then $\tau \in Y^{(*)}$; then $Y_{M'} \subseteq X_M$ and the diagram*

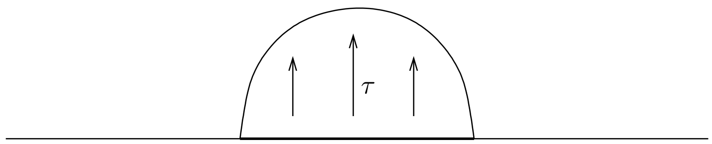
$$\begin{array}{ccc} Y & \xrightarrow{f_{M'}} & Y_{M'} \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{f_M} & X_M \end{array}$$

is commutative, where M' is the restriction of M to G_Y. The CW complex X_M is called Morse complex of X with respect to the acyclic matching M.

Sketch of proof. The compatibility with a compact grading is used to deal with infinite matchings. Here we make the further assumption that X is a finite CW complex, and such compatibility condition becomes unnecessary.

We prove the statement by induction on the number of cells of X. If X has only one cell there is nothing to prove (M must be empty). Consider then the general case. Denote by $\leq_M$ the partial order on $X^{(*)}$ associated to the acyclic graph G_X^M. Let σ be a $\leq_M$-maximal cell, and set $n = \dim \sigma$. There are two cases.

- The cell σ is not M-essential. Then there exists some other cell τ, of dimension $n+1$, such that $(\tau \to \sigma) \in M$. Since σ is $\leq_M$-maximal, it is not a face of any cell other than τ. Moreover, σ is a regular face of τ. Then it is possible to collapse the cell σ on τ as in the following figure.

  σ

  We obtain a complex X' homotopy equivalent to X and with two less cells (σ and τ, which were non-essential), so the induction hypothesis applies.

- The cell σ is M-essential. Then by maximality it isn't face of any other cell of X, thus $X = X' \cup_\varphi D^n$. By induction, X' is homotopy equivalent to its Morse complex X'_M through a homotopy equivalence $f'_M \colon X' \to X'_M$. Construct X_M by attaching D^n to X'_M via the attaching map $f'_M \circ \varphi$. Then $f'_M \colon X' \to X'_M$ extends to a homotopy equivalence $f_M \colon X \to X_M$. □

Consider again the case where X is the torus of Figure 3.1. The acyclic matching M on the right of Figure 3.2 gives rise to a Morse complex X_M with one 2-cell (corresponding to A), two 1-cells (corresponding to a and b) and one 0-cell (corresponding to x). More explicitly, the procedure described in the proof of Theorem 3.12 says to remove the cell A (which is the only $\leq_M$-maximal cell), collapse the cells B and c, and then attach A again. This is shown in Figure 3.3. Notice that a CW structure for a torus cannot have less than one 2-cell, two 1-cells and one 0-cell, since the homology groups have rank 1, 2 and 1 respectively. Indeed, the Morse complex X_M is a minimal CW complex, in the sense that the number of k-cells is exactly $\operatorname{rk} H_k(X_M)$ for all $k \in \mathbb{N}$.

Remark 3.13. It is not true in general that a CW complex X admits an acyclic matching M such that X_M is a minimal CW complex, since in this case the homology of X would be torsion-free.

We are finally going to prove a lemma which will be useful in the next chapters, when it will come to apply discrete Morse theory.

Lemma 3.14. *Let M be a matching on X and let η be a P-grading on X compatible with M. Let M_p be the restriction of M to the fiber $\eta^{-1}(p)$, for all $p \in P$. If M_p is acyclic for all $p \in P$, then M is also acyclic.*

Proof. Suppose by contradiction that the graph G_X^M contains a cycle. Since the edges in M increase the dimension by 1 whereas all the others lower the

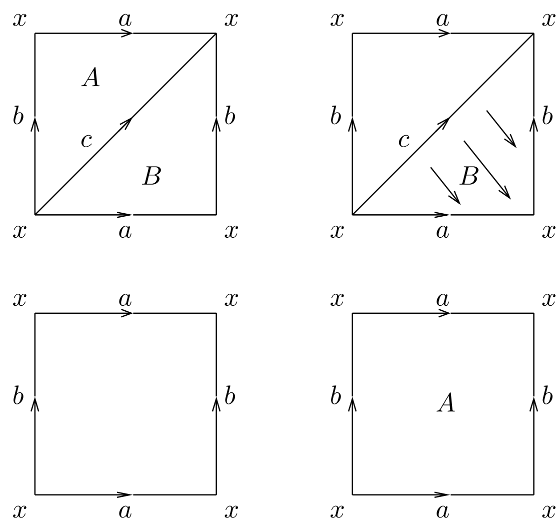


Figure 3.3: Morse collapses for a torus.

dimension by 1, a cycle must be of the form

$$\tau_1 \longrightarrow \sigma_1 \xrightarrow{M} \tau_2 \longrightarrow \sigma_2 \xrightarrow{M} \ldots \longrightarrow \sigma_{k-1} \xrightarrow{M} \tau_k \longrightarrow \sigma_k \xrightarrow{M} \tau_1,$$

where the edges labelled with M are those belonging to M. Since $\tau_i \geq \sigma_i$ in $X^{(*)}$ we have that $\eta(\tau_i) \geq \eta(\sigma_i)$ in P, for all $i = 1, \ldots, k$. Moreover $\eta(\sigma_i) = \eta(\tau_{i+1})$ since $(\tau_{i+1} \to \sigma_i) \in M$, for all $i = 1, \ldots, k$ (the indices are taken modulo k). Therefore

$$\eta(\tau_1) \geq \eta(\sigma_1) = \eta(\tau_2) \geq \eta(\sigma_2) = \cdots = \eta(\tau_k) \geq \eta(\sigma_k) = \eta(\tau_1).$$

The first and the last term of this chain of inequalities are equal, so all the terms are equal to the same element $p \in P$. Then this cycle is contained in the graph $G_X^{M_p}$ and therefore M_p is not acyclic, which is a contradiction. $\square$

In view of Lemma 3.14, it possible to weaken the hypothesis of Theorem 3.12 by removing the requirement of M being acyclic and asking instead that M_p is acyclic for all $p \in P$ (where M_p is the restriction of M to the fiber $\eta^{-1}(p)$). In this way the P-grading η is used to obtain both compactness and acyclicity.

Chapter 4

The $K(\pi, 1)$ conjecture

The $K(\pi, 1)$ conjecture for Artin groups states that a certain topological space constructed from the dual representation of a Coxeter group is a classifying space for the corresponding Artin group. Such conjecture has interesting consequences, and was proved only for some families of Artin groups (first of all, for Artin groups of finite type). In this chapter we are going to present the $K(\pi, 1)$ conjecture, together with some of its consequences and a proof of the conjecture in the case of groups of finite type. Such proof, different from the first one by Deligne [Del72], partially follows the line of Paris [Par12] but has major simplifications thanks to the use of discrete Morse theory. We will mainly follow the notations of [Par12].

4.1 Statement of the conjecture

Let (Γ, S) be a Coxeter graph, and let W_Γ and A_Γ be the corresponding Coxeter and Artin groups. Recall from Section 1.7 that W_Γ acts on the Tits cone $I \subseteq V^*$. The union of the regular orbits of such action is the complement in I of a family $\mathcal{A}$ of hyperplanes of V^*. Define the topological space

$$N(\Gamma) = (I \times I) \setminus \left(\bigcup_{H \in \mathcal{A}} H \times H \right),$$

on which W_Γ acts freely and properly discontinuously by Corollary 1.29. Define then the quotient space

$$\bar{N}(\Gamma) = N(\Gamma)/W_\Gamma,$$

and notice that the projection $N(\Gamma) \to \bar{N}(\Gamma)$ is a covering map. As we will see, the fundamental group of $\bar{N}(\Gamma)$ is canonically isomorphic to the Artin group A_Γ. The $K(\pi, 1)$ conjecture, due to Brieskorn [Bri73] (for groups of finite type), Arnold, Pham, and Thom [vdL83] (in full generality), is the following.

Conjecture 4.1 ($K(\pi, 1)$ conjecture)**.** The space $\bar{N}(\Gamma)$ is a classifying space for the Artin group A_Γ.

Recall that a classifying space for a group π is a connected topological space with fundamental group isomorphic to π and with trivial higher homotopy groups. For spaces with the homotopy type of a CW complex, having trivial higher homotopy groups is equivalent to having a contractible universal cover [Hat02]. A classifying space for a group π is usually called a $K(\pi,1)$, which gives the name to the conjecture.

We now give a few definitions which will be needed to present the current state of the $K(\pi,1)$ conjecture.

Definition 4.2. A Coxeter graph (Γ, S) is said to be *free of infinity* if $m_{s,t} \neq \infty$ for all $s,t \in S$.

Definition 4.3. Denote by S^f and $S^{<\infty}$ the subsets of the power set $\mathcal{P}(S)$ defined by

$$\begin{aligned} S^f &= \{T \subseteq S \mid \Gamma|_T \text{ is of finite type}\}; \\ S^{<\infty} &= \{T \subseteq S \mid \Gamma|_T \text{ is free of infinity}\}. \end{aligned}$$

Definition 4.4. The *dimension* of a Coxeter graph (Γ, S) is the maximum cardinality of a set $X \in S^f$.

Definition 4.5. A Coxeter graph (Γ, S) is called of *large type* if $m_{s,t} \geq 3$ for all $s,t \in S$ such that $s \neq t$.

Definition 4.6. A Coxeter graph (Γ, S) is called of *FC type* if $S^f = S^{<\infty}$.

Definition 4.7. A Coxeter graph (Γ, S) is of *affine type* if the Tits cone I is an open half-space in V^* (recall the notations of Section 1.7) and the action of W_Γ can be restricted to an affine hyperplane $E \subseteq V^*$ in such a way that each $s \in S$ acts as an orthogonal reflection in E.

We extend all these concepts to the corresponding Coxeter and Artin groups (e.g. an Artin group is said to be of large type if its Coxeter graph is of large type, etc.). So far, the $K(\pi,1)$ conjecture has been proved for the following families of Artin groups.

- Artin groups of finite type [Del72].
- Artin groups of dimension ≤ 2 [CD95]. This family includes Artin groups of large type, for which the $K(\pi,1)$ conjecture was previously proved in [Hen85].
- Artin groups of FC type [CD95].
- Some families of Artin groups of affine type, namely those called $\tilde{A}_n$, $\tilde{B}_n$ and $\tilde{C}_n$ [Oko79, CMS10].

4.2 The Salvetti complex

In the rest of this work, a central role is played by the Salvetti complex of a Coxeter graph (Γ, S). Such complex was first defined by Salvetti [Sal87] for Coxeter graphs of finite and affine type, and later generalized by for arbitrary Coxeter graphs (see [Sal94, Par12]). We are going to define the Salvetti complex as in [Par12], and we will quote some known results about it.

Definition 4.8. Given a poset $(P, \leq)$, its *derived complex* is a simplicial complex with P as set of vertices and having a simplex $\{p_1, \dots, p_k\}$ for every chain $p_1 \leq p_2 \leq \dots \leq p_k$ in P.

Definition 4.9. Let $T \subseteq S$. An element $w \in W$ is T*-minimal* if it is the unique element of smallest length in the coset wW_T (cf. Corollary 1.25).

Consider now the set $W \times S^f$, with the following partial order: $(u, T) \leq (v, R)$ if $T \subseteq R$, $v^{-1}u \in W_R$ and $v^{-1}u$ is T-minimal.

Lemma 4.10. The relation $\leq$ defined above is indeed a partial order relation on $W \times S^f$.

Proof. The reflexive property is obvious. Concerning the antisymmetric property, suppose $(u, T) \leq (v, R)$ and $(v, R) \leq (u, T)$. Then $T \subseteq R \subseteq T$ so $T = R$. Moreover $v^{-1}u$ is T-minimal and also contained in W_T, thus $v^{-1}u = 1$. This means that $(u, T) = (v, R)$. We finally verify the transitive property. Suppose $(u, T) \leq (v, R)$ and $(v, R) \leq (w, Q)$. Then $T \subseteq R \subseteq Q$, so $T \subseteq Q$. Since $v^{-1}u \in W_R$ and $w^{-1}v \in W_Q$, we also have that $w^{-1}u = w^{-1}vv^{-1}u \in W_Q$. Furthermore $v^{-1}u$ is T-minimal and $w^{-1}v$ is R-minimal. By Proposition 1.24, we have that for any $x \in W_x$

$$\begin{aligned}\ell(w^{-1}ux) &= \ell(w^{-1}vv^{-1}ux) = \ell(w^{-1}v) + \ell(v^{-1}ux) \\ &= \ell(w^{-1}v) + \ell(v^{-1}u) + \ell(x) = \ell(w^{-1}vv^{-1}u) + \ell(x) \\ &= \ell(w^{-1}u) + \ell(x) \geq \ell(w^{-1}u).\end{aligned}$$

This means that $w^{-1}u$ is T-minimal. □

Lemma 4.11 ([Par12])**.** Let $(u, T) \in W \times S^f$, and set

$$\begin{aligned}P &= \{(v, R) \in W \times S^f \mid (v, R) \leq (u, T)\}, \\ P_1 &= \{(v, R) \in W \times S^f \mid (v, R) < (u, T)\}.\end{aligned}$$

Call P' and P_1' the geometric realizations of the derived complexes of $(P, \leq)$ and $(P_1, \leq)$, respectively. Then the pair (P', P_1') is homeomorphic to the pair (D^n, S^{n-1}) for $n = |T|$.

Definition 4.12. The *Salvetti complex* of a Coxeter graph Γ, denoted by $\mathrm{Sal}(\Gamma)$, is the geometric realization of the derived complex of $(W \times S^f, \leq)$. By Lemma 4.11 it has a CW structure with one cell $C(u, T)$ for all $(u, T) \in W \times S^f$, where the dimension of a cell $C(u, T)$ is $|T|$.

Notice that the dimension of the complex $\mathrm{Sal}(\Gamma)$ coincides with the dimension of the corresponding Coxeter graph Γ, as defined in Section 4.1.

The Coxeter group W acts on $W \times S^f$ by left-multiplication on the first coordinate, and thus also acts on $\mathrm{Sal}(\Gamma)$. Such action is free, properly discontinuous and cellular, so the quotient map

$$\mathrm{Sal}(\Gamma) \to \mathrm{Sal}(\Gamma)/W$$

is a covering map. Moreover such covering map induces a CW structure on the quotient space $\overline{\mathrm{Sal}}(\Gamma) = \mathrm{Sal}(\Gamma)/W$. The complex $\overline{\mathrm{Sal}}(\Gamma)$ has one cell $\bar{C}(T)$ of dimension $|T|$ for each $T \in S^f$.

The reason for which we are interested in studying the Salvetti complex is that it is a CW model for the space $N(\Gamma)$, as is stated in the following theorem.

Theorem 4.13 ([Sal87])**.** There exists a W-equivariant homotopy equivalence $\mathrm{Sal}(\Gamma) \to N(\Gamma)$, which induces a homotopy equivalence $\overline{\mathrm{Sal}}(\Gamma) \to \bar{N}(\Gamma)$.

Let's describe in more detail the combinatorics of the low-dimensional cells of the complexes $\mathrm{Sal}(\Gamma)$ and $\overline{\mathrm{Sal}}(\Gamma)$.

- The 0-cells of $\mathrm{Sal}(\Gamma)$ are in one-to-one correspondence with the elements of the Coxeter group W. For this reason we will often denote a 0-cell by w instead of $C(w, \varnothing)$.

- Since $\{s\} \in S^f$ for all $s \in S$, we have a 1-cell $C(w, \{s\})$ joining vertices w and ws for each $w \in W$ and $s \in S$. Notice that the 1-cell $C(ws, \{s\})$ joins vertices w and ws, but is different from $C(w, \{s\})$. Orient the 1-cell $C(w, \{s\})$ from w to ws.

- A 2-cell $C(w, \{s,t\})$ exists only if $\{s,t\} \in S^f$, i.e. if $m = m_{s,t} \neq \infty$. If it exists, such 2-cell is a $2m$-agon with vertices

  $$\begin{aligned}&w,\, ws,\, wst,\, \dots,\, w\,\Pi(s,t,m-1),\, w\,\Pi(s,t,m) = w\,\Pi(t,s,m),\\ &w\,\Pi(t,s,m-1),\, \dots,\, wt.\end{aligned}$$

  See also Figure 4.1 for a representation of such cell in the case $m = 3$.

The quotient complex $\overline{\mathrm{Sal}}(\Gamma)$ has one 0-cell $\bar{C}(\varnothing)$, a 1-cell $\bar{C}(\{s\})$ for each $s \in S$, and a 2-cell $\bar{C}(\{s,t\})$ for each $\{s,t\} \in S^f$. Therefore the fundamental group of $\overline{\mathrm{Sal}}(\Gamma)$ admits a representation with a generator σ_s for each $s \in S$ and a relation for each 2-cell $\bar{C}(\{s,t\})$. Such relation turns out to be exactly of the form

$$\Pi(\sigma_s, \sigma_t, m_{s,t}) = \Pi(\sigma_t, \sigma_s, m_{s,t}),$$

so we have the following result.

Theorem 4.14 ([vdL83])**.** The fundamental group of $\bar{N}(\Gamma)$ is isomorphic to the Artin group A_Γ.

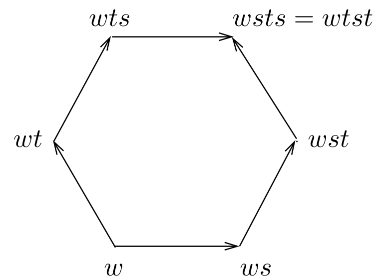


Figure 4.1: Example of a 2-cell $C(w, \{s,t\})$ of the complex $\mathrm{Sal}(\Gamma)$, in the case $m_{s,t} = 3$.

For a CW complex, having trivial homotopy groups is equivalent to being contractible. Therefore Theorems 4.13 and 4.14 allow us to reformulate the $K(\pi, 1)$ conjecture as follows.

Conjecture 4.15. The universal cover of both $\overline{\mathrm{Sal}}(\Gamma)$ and $\mathrm{Sal}(\Gamma)$ is contractible.

Another consequence of Theorem 4.13 is the following: if the $K(\pi, 1)$ conjecture holds for an Artin group A_Γ, then homology and cohomology of A_Γ are trivial in dimension higher than the dimension of A_Γ, and in particular A_Γ is torsion-free. It is worth mentioning that it is not known in general if Artin groups are torsion-free.

Now that we have CW models for $N(\Gamma)$ and $\bar{N}(\Gamma)$, we can apply discrete Morse theory in order to get some information about their homotopy types and about the validity of the $K(\pi, 1)$ conjecture. To do so, we first define a CW structure on the universal cover of the Salvetti complex.

Consider the set $A_\Gamma \times S^f$ together with the partial order $\leq$ defined as follows: $(\alpha, T) \leq (\beta, R)$ if $T \subseteq R$ and α can be written as $\alpha = \beta\tau(w)$ for some T-minimal element $w \in W_R$. Lemmas 4.10 and 4.11 have analogs for $A_\Gamma \times S^f$. Define then $\widetilde{\mathrm{Sal}}(\Gamma)$ as the geometric realization of the derived complex of $(A_\Gamma \times S^f, \leq)$, with the natural CW structure having a cell $\tilde{C}(\alpha, T)$ of dimension $|T|$ for each $(\alpha, T) \in A_\Gamma \times S^f$.

Proposition 4.16 ([Par12])**.** The projections $\widetilde{\mathrm{Sal}}(\Gamma) \to \mathrm{Sal}(\Gamma)$ and $\widetilde{\mathrm{Sal}}(\Gamma) \to \overline{\mathrm{Sal}}(\Gamma)$, induced by the projections $A_\Gamma \times S^f \to W \times S^f$ and $A_\Gamma \times S^f \to S^f$, are cellular covering maps. Moreover the complex $\widetilde{\mathrm{Sal}}(\Gamma)$ is simply connected, and therefore is the universal cover of both $\mathrm{Sal}(\Gamma)$ and $\overline{\mathrm{Sal}}(\Gamma)$.

The Artin group A_Γ naturally acts on the complex $\widetilde{\mathrm{Sal}}(\Gamma)$, similarly to how the Coxeter group W_Γ acts on $\mathrm{Sal}(\Gamma)$, and this action is free, properly discontinuous and cellular.

Notice finally that both the complexes $\mathrm{Sal}(\Gamma)$ and $\widetilde{\mathrm{Sal}}(\Gamma)$ are regular (in the sense of Definition 3.3). On the other hand $\overline{\mathrm{Sal}}(\Gamma)$ is not regular since, for instance, all the 1-cells are attached to the same 0-cell.

4.3 The $K(\pi,1)$ conjecture for Artin groups of finite type

The first proof of the $K(\pi,1)$ conjecture in the case of Artin groups of finite type is due to Deligne [Del72]. Quite recently, Paris suggested a different proof for this result which is based on the combinatorial constructions of Section 4.2 [Par12]. The aim of this section is to present a new proof, which partly follows the one of Paris but is, in our opinion, simpler and more understandable. It relies on discrete Morse theory.

As in [Par12] we are going to define a subcomplex $\widetilde{\mathrm{Sal}}^+(\Gamma)$ of $\widetilde{\mathrm{Sal}}(\Gamma)$, which is the geometric realization of the derived complex of $(A_\Gamma^+ \times S^f, \leq)$ viewed as subposet of $(A_\Gamma \times S^f, \leq)$. Essentially, $\widetilde{\mathrm{Sal}}^+(\Gamma)$ is the subcomplex of $\widetilde{\mathrm{Sal}}(\Gamma)$ having all the cells $\widetilde{C}(\alpha, T)$ such that α belongs to the positive monoid A_Γ^+.

The key result is the following. Its proof is where our work differs from [Par12]. Notice that for now we still don't need to assume that Γ is of finite type.

Theorem 4.17 (cf. [Par12])**.** The subcomplex $\widetilde{\mathrm{Sal}}^+(\Gamma)$ is contractible.

Proof. Our aim is to construct an acyclic matching on $\widetilde{\mathrm{Sal}}^+(\Gamma)$ having only one essential cell (in dimension 0), and then apply Theorem 3.12.

Set $X = \widetilde{\mathrm{Sal}}^+(\Gamma)$. Define a function $\eta\colon X^{(*)} \to \mathbb{N}$ as follows:

$$\eta\big(\tilde{C}(\alpha, T)\big) = \max_{w \in W_T} \ell\big(\alpha\tau(w)\big).$$

We are now going to verify that η is a compact $\mathbb{N}$-grading on X. Suppose that $(\alpha, T) \leq (\beta, R)$ in $X^{(*)}$. This means that the same relation holds in $A_\Gamma^+ \times S^f$, thus $T \subseteq R$ and α can be written as $\alpha = \beta\tau(u)$ for some T-minimal element $u \in W_R$. Therefore

$$\begin{aligned}
\eta\big(\tilde{C}(\alpha, T)\big) &= \max_{w \in W_T} \ell\big(\alpha\tau(w)\big) \\
&= \ell(\alpha) + \max_{w \in W_T} \ell(w) \\
&= \ell(\beta) + \ell(u) + \max_{w \in W_T} \ell(w) \\
&= \ell(\beta) + \max_{w \in W_T} \ell(uw) \\
&\leq \ell(\beta) + \max_{v \in W_R} \ell(v) \\
&= \eta\big(\tilde{C}(\beta, R)\big),
\end{aligned}$$

where the fourth equality follows from Proposition 1.24 since u is T-minimal. We still need to see that η is a compact grading. If a cell $c = \tilde{C}(\alpha, T)$ is such that $\eta(c) \leq n$ for some fixed $n \in \mathbb{N}$, then

$$\ell(\alpha) \leq \max_{w \in W_T} \ell\big(\alpha\tau(w)\big) \leq n.$$

In the Artin monoid A_Γ^+ there is only a finite number of elements of length $\leq n$, so the subcomplex $X_{\leq n}$ has only finitely many cells. Thus it is compact.

We want to construct an acyclic matching on the fibers $\eta^{-1}(n)$ for each $n \in \mathbb{N}$. To do so, we first prove a few intermediate results.

(i) We claim that, given any cell $\tilde{C}(\beta, T)$ of X, there is exactly one 0-cell $\tilde{C}(\alpha, \varnothing)$ lying in the same fiber and such that $\tilde{C}(\alpha, \varnothing) \leq \tilde{C}(\beta, T)$. To prove this, first notice that the relation $\tilde{C}(\alpha, \varnothing) \leq \tilde{C}(\beta, T)$ is true if and only if α can be written in the form $\alpha = \beta\tau(u)$ for some $u \in W_T$. If we add the condition that the two cells must lie in the same fiber, then we have the following inequality:

$$\eta\big(\tilde{C}(\beta, T)\big) = \max_{w \in W_T} \ell(\beta\tau(w)) \geq \ell(\beta\tau(u)) = \eta\big(\tilde{C}(\alpha, \varnothing)\big).$$

The equality holds if and only if u is the unique element of maximal length in W_T. This means that there is exactly one 0-cell which is both in the boundary of $\tilde{C}(\beta, T)$ and in the same fiber.

(ii) Let $\tilde{C}(\alpha, \varnothing)$ be a 0-cell of X. We want to prove that for any $T \in S^f$ there is at most one cell of the form $\tilde{C}(\beta, T)$ in the same fiber of $\tilde{C}(\alpha, \varnothing)$ and such that $\tilde{C}(\alpha, \varnothing) \leq \tilde{C}(\beta, T)$. Indeed, by step (i) we know that the only 0-cell in the boundary of $\tilde{C}(\beta, T)$ and lying in the same fiber is $\tilde{C}(\beta\Delta_T, \varnothing)$, where Δ_T is the fundamental element of A_T^+. So there are two cases: if α is right divisible by Δ_T, then there is exactly one cell of the form $\tilde{C}(\beta, T)$ having $\tilde{C}(\alpha, \varnothing)$ in its boundary and lying in the same fiber (it is obtained setting $\beta = \alpha\Delta_T^{-1}$); otherwise, if α is not right divisible by Δ_T, then there is no 0-cell of the form $\tilde{C}(\beta, T)$ satisfying these two conditions.

(iii) Let $\tilde{C}(\alpha, \varnothing)$ be a 0-cell of X, and let

$$T = \left\{ s \in S \;\middle|\; \begin{array}{l} \text{there exists some } \beta \in A_\Gamma^+ \text{ such that} \\ \tilde{C}(\alpha, \varnothing) \leq C(\beta, \{s\}) \text{ and } \eta\big(\tilde{C}(\alpha, \varnothing)\big) = \eta\big(\tilde{C}(\beta, \{s\})\big) \end{array} \right\}.$$

We want to prove that $T \in S^f$ and that there exists a unique $\gamma \in A_\Gamma^+$ such that

$$\tilde{C}(\alpha, \varnothing) \leq C(\gamma, T) \text{ and } \eta\big(\tilde{C}(\alpha, \varnothing)\big) = \eta\big(\tilde{C}(\gamma, T)\big).$$

By step (ii), α is right multiple of all the elements in $\Sigma_T = \{\sigma_s \mid s \in T\}$; more precisely, using the notation of Section 2.3, $T = I(\alpha)$. By Proposition 2.10 there exists a least right common multiple of Σ_T, which means that $T \in S^f$ (by Theorem 2.19). Moreover, by Lemma 2.20 we have that such least right common multiple is precisely Δ_T, which means that α is right multiple of Δ_T. Again by step (ii), there exists exactly one $\gamma \in A_\Gamma^+$ such that the cell $\tilde{C}(\gamma, T)$ contains the 0-cell $\tilde{C}(\alpha, \varnothing)$ in its boundary and lies in the same fiber.

(iv) Putting together steps (ii) and (iii), we have that any connected component of the subgraph $\eta^{-1}(n) \subseteq G_X$ (see Definition 3.5) has exactly one 0-cell $\tilde{C}(\alpha, \varnothing)$ and is isomorphic to the Hasse diagram of the power set $\mathscr{P}(T)$ where $T = I(\alpha)$. Indeed, for each $R \subseteq T$ there is exactly one cell of the form $\tilde{C}(\beta_R, R)$ which has the 0-cell $\tilde{C}(\alpha, \varnothing)$ in its boundary and lies in the same fiber; moreover, if we have $R \subseteq Q \subseteq T$ then

$$\beta_Q = \alpha \Delta_Q^{-1} = \beta_R \Delta_R \Delta_Q^{-1},$$

thus $\tilde{C}(\beta_R, R) \leq \tilde{C}(\beta_Q, Q)$. In Figure 4.2 are shown the isomorphism types of connected components of $\eta^{-1}(n) \subseteq G_X$ for $|T| = 0, 1, 2, 3$.

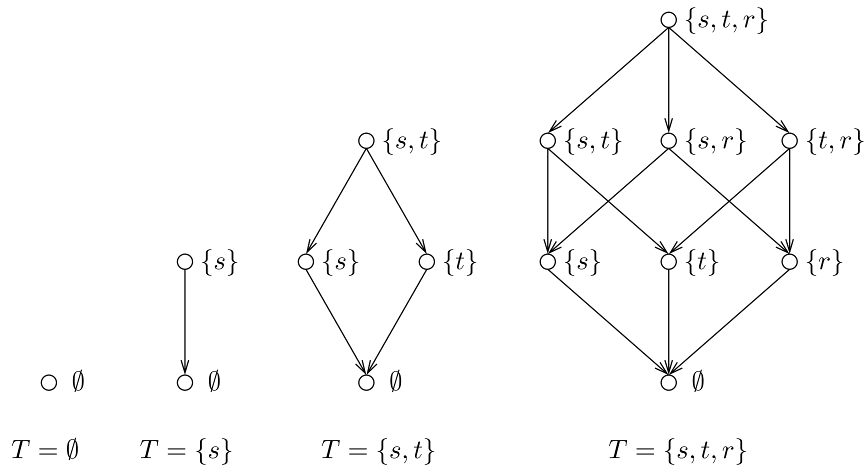


Figure 4.2: Isomorphism types of the connected components of the subgraph $\eta^{-1}(n)$ of G_X, in the cases $|T| = 0, 1, 2, 3$.

We are now able to describe an acyclic matching on the fibers $\eta^{-1}(n)$. Fix a connected component $\mathscr{C}$ of a fiber $\eta^{-1}(n) \subseteq G_X$. As proved in (iv), $\mathscr{C}$ is isomorphic to the Hasse diagram $\mathscr{H}$ of $\mathscr{P}(T)$ for some $T \in S^f$. Unless $\mathscr{C}$ is the connected component of the 0-cell $\tilde{C}(1, \varnothing)$, it contains at least two cells; this is true because $T = I(\alpha)$ has at least one element for $\alpha \neq 1$. If $|T| \geq 1$, fix an element $s \in T$ and consider the following matching M on $\mathscr{H}$:

$$M = \big\{ (R \cup \{s\} \to R) \mid R \subseteq T \setminus \{s\} \big\}.$$

See Figure 4.3 for a drawing of such matching in the case $|T| = 3$. To see that M is acyclic, consider the following partition of the set of vertices $\mathscr{P}(T)$ of $\mathscr{H}$:

$$V_1 = \{ R \subseteq T \mid s \in R \},$$
$$V_2 = \{ R \subseteq T \mid s \notin R \}.$$

All the edges in M go from V_1 to V_2, whereas all the other edges connect two vertices in the same component. This easily implies that M is acyclic.

Putting together the matchings on each connected component $\mathscr{C} \cong \mathscr{H}$, we finally obtain a matching on $X^{(*)}$ compatible with η. The only essential cell

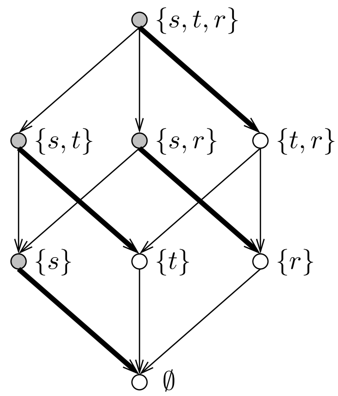


Figure 4.3: The matching M on $\mathscr{H}$ in the case $T = \{s, t, r\}$. The nodes in V_1 are grey, whereas those in V_2 are white.

is the 0-cell $\tilde{C}(1, \varnothing)$. Therefore by Theorem 3.12 the complex X is homotopy equivalent to a CW complex with only one cell (in dimension 0), i.e. X is contractible. $\square$

In Figure 4.4 is shown the subcomplex $\widetilde{\mathrm{Sal}}^+(\Gamma)$ for $S = \{s, t\}$ and $m_{s,t} = 2$, together with the matching used to prove Theorem 4.17.

Thanks to the results of Section 2.2, when Γ is of finite type it is possible to "nicely" cover the entire complex $\widetilde{\mathrm{Sal}}(\Gamma)$ with copies of $\widetilde{\mathrm{Sal}}^+(\Gamma)$. This is how we will then prove that Theorem 4.17 implies the $K(\pi, 1)$ conjecture for Artin groups of finite type.

Proposition 4.18 ([Par12])**.** Let Γ be a Coxeter graph of finite type. Then there exists an infinite chain $Y_0 \subseteq Y_1 \subseteq \dots$ of subcomplexes of $\widetilde{\mathrm{Sal}}(\Gamma)$ such that

$$\widetilde{\mathrm{Sal}}(\Gamma) = \bigcup_{i \in \mathbb{N}} Y_i$$

and each Y_i is isomorphic (as a CW complex) to $\widetilde{\mathrm{Sal}}^+(\Gamma)$.

Proof. Let Δ be the fundamental element of A_Γ^+ (see Section 2.2). Define then the subcomplexes Y_i as

$$Y_i = \Delta^{-i}\, \widetilde{\mathrm{Sal}}^+(\Gamma).$$

By Theorem 2.16, any element $\alpha \in A_\Gamma$ can be written in the form $\Delta^{-i}\beta$ for some $i \in \mathbb{N}$ and $\beta \in A_\Gamma^+$. This means that any cell $\tilde{C}(\alpha, T)$ of $\widetilde{\mathrm{Sal}}(\Gamma)$ is contained in some subcomplex Y_i, so the union of such subcomplexes covers all $\widetilde{\mathrm{Sal}}(\Gamma)$. $\square$

Recall now the following classical result, which be deduced from [Hat02], Corollary 4G.3.

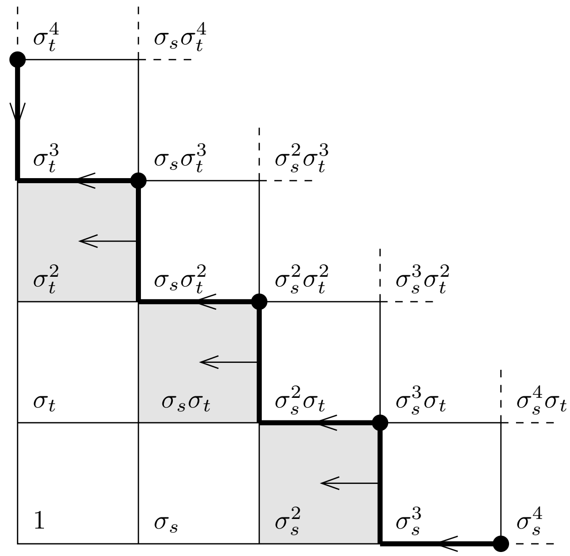


Figure 4.4: The subcomplex $\widetilde{\mathrm{Sal}}^+(\Gamma)$ for $S = \{s, t\}$ and $m_{s,t} = 2$, with the matching on $\eta^{-1}(4)$.

Lemma 4.19. Let X be a CW complex, and let $\{Y_i \mid i \in \mathbb{N}\}$ be a family of contractible subcomplexes of X such that $Y_i \subseteq Y_{i+1}$ for all $i \in \mathbb{N}$ and

$$\bigcup_{i\in\mathbb{N}} Y_i = X.$$

Then X is also contractible.

Proof. The image of any map $S^n \to X$ is compact, so it is contained in a finite subcomplex of X. Thus it is contained in Y_i for some i. Since Y_i is contractible, the map itself is homotopic to a constant map in Y_i. This proves that all the homotopy groups of X are trivial. Then the inclusion $\{x\} \hookrightarrow X$ for any $x \in X$ is a weak homotopy equivalence, so by Whitehead's theorem it is also a homotopy equivalence. □

We are finally able to prove the $K(\pi, 1)$ conjecture for Artin groups of finite type.

Theorem 4.20 ([Del72, Par12])**.** Let Γ be a Coxeter graph of finite type. Then $\bar{N}(\Gamma)$ is a classifying space for the corresponding Artin group A_Γ.

Proof. We have already seen that it is enough to prove that the complex $\widetilde{\mathrm{Sal}}(\Gamma)$ is contractible. This follows from Lemma 4.19 using the family $\{Y_i\}$ of subcomplexes given by Proposition 4.18, each of them being contractible by Theorem 4.17. □

Chapter 5

Classifying space of Artin monoids

In this final chapter we will present some relations between the $K(\pi, 1)$ conjecture and the notion of classifying spaces of monoids introduced in [Seg73]. In the first section we are going to introduce classifying spaces of monoids, particularly in the case of monoids which inject into their groupification (this is the case we are interested in, since an Artin monoid injects into the corresponding Artin group). The second section is devoted to an alternative proof of a theorem by Dobrinskaya [Dob06], which gives a reformulation of the $K(\pi, 1)$ conjecture. Such proof is new (although part of it relies on a work by Ozornova [Ozo13]) and uses the technique of discrete Morse theory.

5.1 Classifying space of monoids

First we are going to introduce the notion of classifying space of a monoid, as a particular case of the classifying space of a small category (viewing a monoid as a category with one object) [Seg73].

Definition 5.1. The classifying space BM of a monoid M is the geometric realization of the following simplicial set. The n-simplices are given by the sequences $(x_1, \ldots, x_n)$ of n elements of M, and are denoted by the symbol $[x_1|\ldots|x_n]$. The face maps send an n-simplex $[x_1|\ldots|x_n]$ to the simplices $[x_2|\ldots|x_n]$, $[x_1|\ldots|x_i x_{i+1}|\ldots|x_n]$ for $i = 1, \ldots, n-1$, and $[x_1|\ldots|x_{n-1}]$. The degeneracy maps send $[x_1|\ldots|x_n]$ to $[x_1|\ldots|x_i|1|x_{i+1}|\ldots|x_n]$ for $i = 0, \ldots, n$.

As shown in [Mil57], the geometric realization of a simplicial set is a CW complex having a n-cell for each non-degenerate n-simplex. Therefore the classifying space of a monoid is a CW complex having as n-cells the simplices $[x_1|\ldots|x_n]$ with $x_i \neq 1$ for all i. Notice also that BM has only one 0-cell denoted by $[\,]$.

Definition 5.2. The groupification of a monoid M is a group G together with a homomorphism $M \to G$ satisfying the following universal property: for any group H and homomorphism $M \to H$, there exists a unique homomorphism $G \to H$ which makes the following diagram commutative.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & G \arrow[dl, dashed] \\ H & \end{tikzcd}$$

Remark 5.3. If any presentation of M is given, then the groupification G of M is the group with the same presentation.

Remark 5.4. The fundamental group of the classifying space BM of a monoid M is given by the groupification G of M. This can be easily seen using the well-known presentation of the fundamental group of a CW complex with one 0-cell: generators are given by the 1-cells, and relations are given by the attaching maps of the 2-cells. In our case the generator set is $\{[x] \mid x \in M,\, x \neq 1\}$ and the relation corresponding to the 2-cell $[x|y]$ is given by $[x][y][xy]^{-1}$ if $xy \neq 1$ and $[x][y]$ if $xy = 1$. This is indeed a presentation for the groupification G of M, by Remark 5.3.

Before focusing on the case of Artin monoids, we are going to give an explicit construction for the universal cover of BM for any monoid M that injects in its groupification G (i.e. the natural map $M \to G$ is injective). This construction generalizes the one of Example 1B.7 in [Hat02]. Let EM be the geometric realization of the simplicial set whose n-simplices are given by the $(n+1)$-tuples $[g|x_1|\dots|x_n]$, where $g \in G$ and $x_i \in M$. The i-th face map sends $[g|x_1|\dots|x_n]$ to

$$\begin{cases} [gx_1|x_2|\dots|x_n] & \text{for } i = 0; \\ [g|x_1|\dots|x_ix_{i+1}|\dots|x_n] & \text{for } 1 \leq i \leq n-1; \\ [g|x_1|\dots|x_{n-1}] & \text{for } i = n. \end{cases}$$

The i-th degeneracy map sends $[g|x_1|\dots|x_n]$ to $[g|x_1|\dots|x_i|1|x_{i+1}|\dots|x_n]$ for all $i = 0, \dots, n$. Notice that the vertices of EM are in bijection with the group G, and that the vertices of an n-simplex $[g|x_1|\dots|x_n]$ are of the form $[gx_1 \cdots x_i]$ for $i = 0, \dots, n$. The group G acts freely and simplicially on EM by left multiplication: an element $h \in G$ sends the simplex $[g|x_1|\dots|x_n]$ to the simplex $[hg|x_1 \dots |x_n]$. Thus the quotient map $EM \to EM/G$ is a covering map.

Lemma 5.5. *EM/G is naturally homeomorphic to BM.*

Proof. A simplex $[x_1|\dots|x_n]$ of BM can be identified with the equivalence class of the simplex $[1|x_1|\dots|x_n]$ of EM/G. This identification is bijective and respects face maps and degeneracy maps, so it is a homeomorphism. □

Proposition 5.6. *The space EM is the universal cover of BM, with the natural covering map $p\colon EM \to BM$ obtained composing the quotient map $EM \to EM/G$ and the homeomorphism $EM/G \to BM$ of Lemma 5.5.*

Proof. We have already seen that p is indeed a covering map. Therefore it is enough to show that EM is simply connected. Choose $[\,]$ and $[1]$ as basepoints for BM and EM respectively, so that $p\colon (EM,*) \to (BM,*)$ becomes a basepoint-preserving covering map. An element c of $\pi_1(BM,*)$ can be represented as a signed sequence $(\epsilon_1[x_1],\dots,\epsilon_k[x_k])$ of 1-cells, where the sign $\epsilon_i = \pm 1$ indicates whether the arc $[x_i]$ is travelled in positive or negative direction. If we lift such path to the covering space EM, we obtain a path passing through the vertices $[1],[x_1^{\epsilon_1}],[x_1^{\epsilon_1}x_2^{\epsilon_2}],\dots,[x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_k^{\epsilon_k}]$. Notice that under the isomorphism $\pi_1(BM,*) \cong G$ of Remark 5.4 the path c corresponds precisely to $x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_k^{\epsilon_k}$. This means that if c is non-trivial in $\pi_1(BM,*)$ then it lifts to a non-closed path in EM. Since $p_*\colon \pi_1(EM,*) \to \pi_1(BM,*)$ is injective, we can conclude that $\pi_1(EM,*)$ is trivial. □

The space EM has a particular subcomplex E^+M consisting of all the cells $[g|x_1|\dots|x_n]$ such that $g \in M$. In analogy to the case when M is a group (for which $EM = E^+M$), we prove that E^+M is contractible.

Proposition 5.7. The space E^+M deformation retracts onto its vertex $[1]$. In particular it is contractible.

Proof. Any simplex $[g|x_1|\dots|x_n]$ of E^+M is a face of the (possibly degenerate) simplex $[1|g|x_1|\dots|x_n]$, which is also in E^+M. Then we have a deformation retraction of E^+M onto the vertex $[1]$ which slides any point $q \in [g|x_1|\dots|x_n]$ along the line segment from q to $[1]$. Such line segment exists in $[1|g|x_1|\dots|x_n]$, and is well defined in E^+M because the attaching maps of the simplices are linear. □

5.2 Dobrinskaya's theorem

In [Dob06], Dobrinskaya proved that the $K(\pi,1)$ conjecture can be reformulated as follows.

Theorem 5.8 ([Dob06])**.** The $K(\pi,1)$ conjecture holds for an Artin group A_Γ if and only if the natural map $BA_\Gamma^+ \to BA_\Gamma$ is a homotopy equivalence.

Dobrinskaya's theorem is particularly interesting since it relates the $K(\pi,1)$ conjecture to the problem of determining when the natural map $M \to G$ between a monoid and its groupification induces a homotopy equivalence $BM \to BG$ between the corresponding classifying spaces. Such phenomenon is known to happen in some cases (see [MS76]), but the general problem is still open.

To prove Theorem 5.8, Dobrinskaya also proved the following result.

Theorem 5.9 ([Dob06])**.** The space $\bar{N}(\Gamma)$ is homotopy equivalent to the classifying space BA_Γ^+ of the Artin monoid A_Γ^+.

It is quite easy to deduce Theorem 5.8 from Theorem 5.9. Indeed, if the natural map $BA_\Gamma^+ \to BA_\Gamma$ is a homotopy equivalence then

$$\bar{N}(\Gamma) \simeq BA_\Gamma^+ \simeq BA_\Gamma,$$

so the $K(\pi, 1)$ conjecture holds for the Artin group A_Γ. On the other hand, if the $K(\pi, 1)$ conjecture holds for A_Γ then both spaces BA_Γ and BA_Γ^+ are classifying spaces for A_Γ, so the natural map $BA_\Gamma^+ \to BA_\Gamma$ must be a homotopy equivalence since it induces an isomorphism at the level of fundamental groups.

In the rest of this section we will present a new proof of Theorem 5.9 based on discrete Morse theory. Some ideas are taken from a recent work of Ozornova [Ozo13], but we will prove the stronger statement that the space BA_Γ^+ can be collapsed (in the sense of discrete Morse theory) to obtain a CW complex which is naturally homotopy equivalent to the Salvetti complex $\overline{\mathrm{Sal}}(\Gamma)$. This in particular answers some of the questions left open in [Ozo13], Section 7.

From now on, let (Γ, S) be a Coxeter graph. When Γ is of finite type we are able to show that EA_Γ^+ is contractible, using an argument similar to the one used to prove that $\widetilde{\mathrm{Sal}}(\Gamma)$ is contractible.

Theorem 5.10. If Γ is a Coxeter graph of finite type, then the space EA_Γ^+ is contractible.

Proof. To make the notation more readable denote the space EA_Γ^+ by X and its subcomplex $E^+A_\Gamma^+$ by X^+. For any $h \in A_\Gamma$ the subcomplex hX^+ is homeomorphic to X^+, and therefore it is contractible by Proposition 5.7. Notice also that hX^+ consists of all the simplices $[g|x_1|\dots|x_n]$ of X such that $g \succeq_{\mathrm{L}} h$. Then, if Δ is the fundamental element of A_Γ^+, X is the union of the subcomplexes $Y_i = \Delta^{-i}X^+$ for $i \in \mathbb{N}$. Since $Y_i \subseteq Y_{i+1}$ for all i, we can apply Lemma 4.19 to conclude that X is contractible. $\square$

Corollary 5.11. If Γ is a Coxeter graph of finite type, then the classifying space BA_Γ^+ is a classifying space for A_Γ.

Proof. This result follows immediately by Remark 5.4, Proposition 5.6 and Theorem 5.10. $\square$

We are going to construct an acyclic matching M on BA_Γ^+ which is essentially a combination of the two matchings used in [Ozo13], with the difference that ours will be entirely on the topological level. Set $Z = BA_\Gamma^+$ and

$$\mathcal{D} = \{\Delta_T \mid T \in S^f \setminus \{\varnothing\}\},$$

where Δ_T is the fundamental element of $A_T^+ \subseteq A_\Gamma^+$. First we are going to describe some definitions and results of [Ozo13], which will lead to the construction of two matchings M_1 and M_2.

- A cell $c = [x_1|\dots|x_n] \in Z^{(*)}$ is μ_1-essential if the product $x_k \cdots x_n$ lies in $\mathcal{D}$ for $1 \leq k \leq n$.
- The μ_1-depth of a cell $c = [x_1|\dots|x_n]$ is given by

  $$d_1(c) = \min\{j \mid [x_j|\dots|x_n] \text{ is } \mu_1\text{-essential}\},$$

  with the convention that $d_1(c) = n+1$ if no such j exists. Notice that c is μ_1-essential if and only if $d_1(c) = 1$.

- For a cell $c = [x_1|\dots|x_n]$ of μ_1-depth d, and for $d \le k \le n$, define $I_k \subseteq S$ to be the unique subset of S with the property that $x_k \cdots x_n = \Delta_{I_k}$.
- A cell $c = [x_1|\dots|x_n]$ of μ_1-depth $d > 0$ is μ_1-collapsible if

$$I(x_{d-1}x_d \cdots x_n) = I_d,$$

Lemma 5.12 ([Ozo13])**.** Define

$$M_1 = \left\{ (c_1 \to c_2) \;\middle|\; \begin{array}{l} c_1 = [x_1|\dots|x_n] \in Z^{(*)} \text{ is } \mu_1\text{-collapsible, and} \\ c_2 = [x_1|\dots|x_{d-1}x_d|\dots|x_n] \text{ where } d = d_1(c_1) \end{array} \right\}.$$

Then M_1 is an acyclic matching on Z with essential cells given by the μ_1-essential cells defined above.

To construct the second matching M_2, assume from now on that the set S carries a total order $\le$. Notice that a μ_1-essential cell $c = [x_1|\dots|x_n]$ is completely characterized by the sequence of subsets $I_1 \subset I_2 \subset \cdots \subset I_k$ defined above.

- A μ_1-essential cell $c = [x_1|\dots|x_n]$ is μ_2-essential if, for any $1 \le k \le n$, $I_k \setminus I_{k+1} = \{s_k\}$ and $s_k = \max I_k$.
- The μ_2-depth of a μ_1-essential cell $c = [x_1|\dots|x_n]$ is given by

$$d_2(c) = \min\{j \mid [x_j|\dots|x_n] \text{ is } \mu_2\text{-essential}\},$$

- A μ_1-essential cell $c = [x_1|\dots|x_n]$ of μ_2-depth $d > 0$ is μ_2-collapsible if

$$\max I_{d-1} = \max I_d.$$

Lemma 5.13 ([Ozo13])**.** Define

$$M_2 = \left\{ (c_1 \to c_2) \;\middle|\; \begin{array}{l} c_1 = [x_1|\dots|x_n] \in Z^{(*)} \text{ is } \mu_2\text{-collapsible, and} \\ c_2 = [x_1|\dots|x_{d-1}x_d|\dots|x_n] \text{ where } d = d_2(c_1) \end{array} \right\}.$$

Then M_2 is an acyclic matching on Z with essential cells given by the non-μ_1-essential cells and the μ_2-essential cells.

Consider now the matching $M = M_1 \cup M_2$ on Z. With a slight abuse of notation, define the length of a cell as

$$\ell([x_1|\dots|x_n]) = \ell(x_1 \cdots x_n)$$

for any cell $[x_1|\dots|x_n] \in Z^{(*)}$. Define also a function $\eta\colon Z^{(*)} \to \mathbb{N} \times \{0,1\}$ as follows:

$$\eta(c) = \begin{cases} (\ell(c), 0) & \text{if } c \text{ is } \mu_1\text{-essential;} \\ (\ell(c), 1) & \text{if } c \text{ is not } \mu_1\text{-essential.} \end{cases}$$

Lemma 5.14. The function $\eta\colon Z^{(*)} \to \mathbb{N} \times \{0,1\}$ is a compact grading on Z, if $\mathbb{N} \times \{0,1\}$ is equipped with the lexicographic order.

Proof. First we have to prove that η is a poset map. For this it is enough to prove that, for any cell $c_1 = [x_1|\dots|x_n] \in Z^{(*)}$ and for any cell c_2 which is a face of c_1, $\eta(c_1) \geq \eta(c_2)$. Suppose by contradiction that $\eta(c_1) < \eta(c_2)$ for some cells c_1 and c_2 as above. Since $\ell(c_1) \geq \ell(c_2)$ the only possibility is that $\eta(c_1) = (k,0)$ and $\eta(c_2) = (k,1)$ where $k = \ell(c_1) = \ell(c_2)$. This means in particular that c_1 is μ_1-essential whereas c_2 is not. Since $\ell(c_1) = \ell(c_2)$, the cell c_2 must be of the form

$$c_2 = [x_1|\dots|x_i x_{i+1}|\dots|x_n]$$

for some $i \in \{1,\dots,n-1\}$. Clearly if c_1 is μ_1-essential then also c_2 is, so we obtain a contradiction.

It only remains to prove that $Z_{(n,q)}$ is compact for all $(n,q) \in \mathbb{N} \times \{0,1\}$. This is immediate since $Z_{(n,q)}$ contains only cells of length $\leq n$ and there is only a finite number of them. □

Proposition 5.15. The matching M on Z is acyclic and compatible with the compact grading η.

Proof. First let us prove that M and η are compatible. If $(c_1 \to c_2) \in M_1$ then, by definition of M_1, we have that $\ell(c_1) = \ell(c_2)$ and that both c_1 and c_2 are not μ_1-essential. On the other hand, if $(c_1 \to c_2) \in M_2$, then $\ell(c_1) = \ell(c_2)$ and both c_1 and c_2 are μ_1-essential. In any case we have $\eta(c_1) = \eta(c_2)$, which means that M and η are compatible.

Consider a fiber $\eta^{-1}(n,q)$, for some $(n,q) \in \mathbb{N} \times \{0,1\}$. It cannot simultaneously contain edges in M_1 and edges in M_2, because the value of q determines whether the cells in $\eta^{-1}(n,q)$ must be μ_1-essential or not. Since M_1 and M_2 are acyclic, the restriction of M to $\eta^{-1}(n,q)$ is also acyclic. This is true for all fibers $\eta^{-1}(n,q)$, therefore by Lemma 3.14 we can conclude that M is also acyclic. □

The previous proposition allows to apply Theorem 3.12 to Z, obtaining a smaller CW complex which we will call Y. The essential cells of the matching M are precisely the μ_2-essential cells. Notice that a μ_2-essential cell $c = [x_1|\dots|x_n] \in Z^{(*)}$ is uniquely identified by the set $I(x_1 \cdots x_n) \in S^f$. This means that the cells of Y are in one-to-one correspondence with S^f.

Call e_T the cell of Y corresponding to the set $T \in S^f$. Then $\dim e_T = |T|$, and the attaching map of e_T has image contained in the union of the cells e_R with $R \subsetneq T$. So any subset $\mathcal{F} \subseteq S^f$ which is closed under inclusion (i.e. $R \subseteq T \in \mathcal{F}$ implies $R \in \mathcal{F}$) corresponds to a subcomplex $Y_{\mathcal{F}}$ of Y. In particular this holds when $\mathcal{F} = T^f$ for any $T \subseteq S$.

In a similar way we have subcomplexes $\overline{\mathrm{Sal}}_{\mathcal{F}}(\Gamma)$ of $\overline{\mathrm{Sal}}(\Gamma)$ for all subsets $\mathcal{F}$ of S^f closed under inclusion.

Remark 5.16. The reduced complex Y is natural with respect to inclusion of the set S. Indeed, consider Coxeter graphs (Γ, S) and (Γ', S') such that $S \subseteq S'$ (with a fixed total order on S', which induces a total order on S) and

$\Gamma'|_S = \Gamma$. Then we obtain reduced complexes Y and Y' such that Y can be naturally identified with the subcomplex Y'_{Sf} of Y'. This is true because for any non-degenerate simplex e of the subcomplex $BA^+_\Gamma \subseteq BA^+_{\Gamma'}$ all the faces of e, as well as the matched cell $\mu(e)$, also belong to the subcomplex BA^+_Γ.

Let us recall a few results of homotopy theory which will be used later. For $n \geq 2$ denote by $\pi'_n(X, A, *)$ the quotient group of $\pi_n(X, A, *)$ by the normal subgroup generated by the elements $[\gamma][f] - [f]$ for $[f] \in \pi_n(X, A, *)$ and $[\gamma] \in \pi_1(X, *)$. Furthermore denote by $h' \colon \pi'_n(X, A, *) \to H_n(X, A)$ the homomorphism induced by the Hurewicz homomorphism $h \colon \pi_n(X, A, *) \to H_n(X, A)$. See also [Hat02], page 370.

Lemma 5.17 ([Hat02], Proposition 0.18)**.** If (X_1, A) is a CW pair and we have attaching maps $f, g \colon A \to X_0$ that are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0.

Corollary 5.18. If X is a CW complex and $f, g \colon S^{n-1} \to X$ are two attaching maps of an n-cell e^n that are homotopic, then $X \sqcup_f e^n \simeq X \sqcup_g e^n$ rel X.

Proof. It follows from the previous lemma with $(X_1, A) = (D^n, S^{n-1})$ and $X_0 = X$. □

We are finally ready to prove that the CW complexes Y and $\overline{\mathrm{Sal}}(\Gamma)$ are homotopy equivalent. To do so, we start with the following lemma.

Lemma 5.19. Up to orientation, the boundary curve of a 2-cell $e_{\{s,t\}}$ of Y is given by

$$\begin{cases} \Pi(e_{\{s\}}, e_{\{t\}}, m_{s,t})\ \Pi(e^{-1}_{\{s\}}, e^{-1}_{\{t\}}, m_{s,t}) & \text{if } m_{s,t} \text{ is even;} \\ \Pi(e_{\{s\}}, e_{\{t\}}, m_{s,t})\ \Pi(e^{-1}_{\{t\}}, e^{-1}_{\{s\}}, m_{s,t}) & \text{if } m_{s,t} \text{ is odd.} \end{cases}$$

Proof. By Remark 5.16 it is sufficient to treat the case $S = \{s, t\}$, so that Y consists only of one 0-cell $e_\varnothing$, two 1-cells $e_{\{s\}}, e_{\{t\}}$ and one 2-cell $e_{\{s,t\}}$. Suppose that $s > t$ (in the other case the result is the same but with reversed orientation). Set

$$\Pi'(a, b, k) = \begin{cases} \Pi(a, b, k) & \text{if } k \text{ is even,} \\ \Pi(b, a, k) & \text{if } k \text{ is odd.} \end{cases}$$

Moreover, set

$$\begin{aligned} \xi^k_s &= \Pi'(\sigma_t, \sigma_s, k), \\ \xi^k_t &= \Pi'(\sigma_s, \sigma_t, k). \end{aligned}$$

Essentially ξ^k_s is a product of k alternating elements (σ_s or σ_t) ending with σ_s, and ξ^k_t is the same but ending with σ_t. For example, $\xi^4_s = \sigma_t\sigma_s\sigma_t\sigma_s$. Set also

$m = m_{s,t}$. The cells $e_\varnothing$, $e_{\{s\}}$, $e_{\{t\}}$ and $e_{\{s,t\}}$ correspond to the M-essential cells of Z, which are the following:

$$\begin{aligned} c_\varnothing &= [\,], \\ c_{\{s\}} &= [\sigma_s], \\ c_{\{t\}} &= [\sigma_t], \\ c_{\{s,t\}} &= [\xi_s^{m-1}|\sigma_t]. \end{aligned}$$

If c is a cell of Z, call $M^{<c}$ the restriction of the matching M to the cells that are $< c$ with respect to the partial order induced by the acyclic graph G_Z^M. Since $M^{<c}$ is also an acyclic matching on Z, compatible with the compact grading η, we can consider the complex $Y^{<c}$ obtained collapsing Z along the matching $M^{<c}$. For simplicity, we will call a cell of some $Y^{<c}$ with the same name as the corresponding $M^{<c}$-essential cell in Z.

We want to prove by induction on k the following two assertions:

(i) the boundary curve of the 2-cell $c = [\xi_t^k|\sigma_s]$ in $Y^{<c}$ is

$$\Pi'([\sigma_t], [\sigma_s], k+1)\,[\xi_s^{k+1}]^{-1}$$

for $1 \le k \le m-1$;

(ii) the boundary curve of the 2-cell $c = [\xi_s^k|\sigma_t]$ in $Y^{<c}$ is

$$\Pi'([\sigma_s], [\sigma_t], k+1)\,[\xi_t^{k+1}]^{-1}$$

for $1 \le k \le m-2$.

The base steps are for $k = 1$. Start from case (i). We have $c = [\sigma_t|\sigma_s]$, whose boundary in Z is given by $[\sigma_t][\sigma_s][\sigma_t\sigma_s]^{-1}$. The 1-cells $[\sigma_t]$ and $[\sigma_s]$ are M-essential, whereas the 1-cell $[\sigma_t\sigma_s]$ is matched with c. Therefore all these 1-cells are $M^{<c}$-essential. This means that the boundary of c in $Y^{<c}$ is also given by $[\sigma_t][\sigma_s][\sigma_t\sigma_s]^{-1}$. Case (ii) is similar: we have $c = [\sigma_s|\sigma_t]$, and its boundary in Z is $[\sigma_t][\sigma_s][\sigma_s\sigma_t]^{-1}$. This is also the boundary in $Y^{<c}$, because $[\sigma_s\sigma_t]$ is matched with c (this wouldn't be true for $m = 2$, but such case is excluded by the condition $k \le m-2$).

We want now to prove step k, case (i), for $2 \le k \le m-1$. We have $c = [\xi_t^k|\sigma_s]$, whose boundary in Z is given by $[\xi_t^k][\sigma_s][\xi_s^{k+1}]^{-1}$. The 1-cell $[\sigma_s]$ is M-essential, so it is in particular $M^{<c}$-essential. The 1-cell $[\xi_s^{k+1}]$ is matched with c in M, so it is $M^{<c}$-essential. Finally the 1-cell $[\xi_t^k]$ is matched with $c' = [\xi_s^{k-1}|\sigma_t]$, whose boundary in $Y^{<c'}$ is by induction hypothesis $\Pi'([\sigma_s], [\sigma_t], k)\,[\xi_t^k]^{-1}$. Thus the boundary of c in $Y^{<c}$ is given by

$$\Pi'([\sigma_s], [\sigma_t], k)\,[\sigma_s]\,[\xi_s^{k+1}]^{-1} = \Pi'([\sigma_t], [\sigma_s], k+1)\,[\xi_s^{k+1}]^{-1}.$$

See the left part of Figure 5.1 for a picture of case (i).

We finally want to prove step k, case (ii), for $1 \le k \le m-2$. We have $c = [\xi_s^k|\sigma_t]$, whose boundary in Z is given by $[\xi_s^k][\sigma_t][\xi_t^{k+1}]^{-1}$. The 1-cell $[\sigma_t]$ is

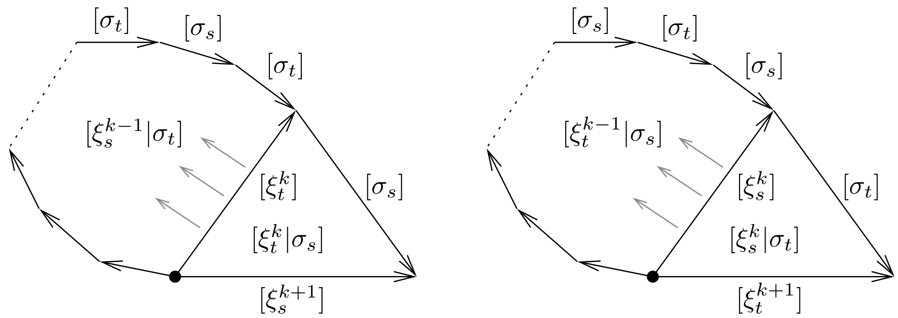


Figure 5.1: On the left: the induction step, case (i); on the right: the induction step, case (ii). The boundary curve for all the 2-cells is denoted clockwise, starting from the black vertex. The light arrows indicate the Morse collapse.

M-essential, so it is $M^{<c}$-essential. The 1-cell $[\xi_t^{k+1}]$ is matched with c in M (notice that this is true only for $k < m-1$, because $\xi_t^m = \xi_s^m$), thus it is also $M^{<c}$-essential. The only 1-cell left to analyze is $[\xi_s^k]$, which is matched with $c' = [\xi_t^{k-1}|\sigma_s]$, whose boundary in $Y^{<c'}$ is by induction $\Pi'([\sigma_t],[\sigma_s],k)\,[\xi_s^k]^{-1}$. Then the boundary of c in $Y^{<c}$ is given by

$$\Pi'([\sigma_t],[\sigma_s],k)\,[\sigma_t]\,[\xi_t^{k+1}]^{-1} = \Pi'([\sigma_s],[\sigma_t],k+1)\,[\xi_t^{k+1}]^{-1}.$$

See the right part of Figure 5.1 for a picture of case (ii).

The induction argument is complete. To end the proof, consider now the 2-cell $c_{\{s,t\}} = [\xi_s^{m-1}|\sigma_t]$ of Z. Its boundary is $[\xi_s^{m-1}][\sigma_t][\xi_t^m]$. The 1-cell $[\sigma_t]$ is M-essential. The 1-cell $[\xi_s^{m-1}]$ is matched with $c' = [\xi_t^{m-2}|\sigma_s]$, whose boundary in $Y^{<c'}$ is $\Pi'([\sigma_t],[\sigma_s],m-1)\,[\xi_s^{m-1}]^{-1}$. The 1-cell $[\xi_t^m] = [\xi_s^m]$ is matched with $c'' = [\xi_t^{m-1}|\sigma_s]$, whose boundary in $Y^{<c''}$ is $\Pi'([\sigma_t],[\sigma_s],m)\,[\xi_s^m]^{-1}$. Therefore the boundary of $c_{\{s,t\}}$ in Y is

$$\begin{aligned}
&\Pi'([\sigma_t],[\sigma_s],m-1)\;[\sigma_t]\;\Big(\Pi'([\sigma_t],[\sigma_s],m)\Big)^{-1}\\
&= \Pi'([\sigma_t],[\sigma_s],m-1)\;[\sigma_t]\;\Pi([\sigma_s]^{-1},[\sigma_t]^{-1},m)\\
&= \Pi'([\sigma_s],[\sigma_t],m)\;\Pi([\sigma_s]^{-1},[\sigma_t]^{-1},m).
\end{aligned}$$

Up to orientation and starting point of the boundary curve, this is exactly what is stated in the lemma.

□

Theorem 5.20. For any Coxeter graph Γ there exists a homotopy equivalence $\psi\colon Y \to \overline{\mathrm{Sal}}(\Gamma)$ such that, for every subset $\mathcal{F}$ of S^f closed under inclusion, the restriction $\psi|_{Y_\mathcal{F}}$ has image contained in $\overline{\mathrm{Sal}}_\mathcal{F}(\Gamma)$ and

$$\psi|_{Y_\mathcal{F}}\colon Y_\mathcal{F} \to \overline{\mathrm{Sal}}_\mathcal{F}(\Gamma)$$

is also a homotopy equivalence.

Proof. We are going to construct simultaneously the map ψ of the statement and its homotopy inverse $\psi' \colon \overline{\mathrm{Sal}}(\Gamma) \to Y$, together with homotopies $F \colon Y \times [0,1] \to Y$ between $\psi' \circ \psi$ and id_Y, and $F' \colon \overline{\mathrm{Sal}}(\Gamma) \times [0,1] \to \overline{\mathrm{Sal}}(\Gamma)$ between $\psi \circ \psi'$ and $\mathrm{id}_{\overline{\mathrm{Sal}}(\Gamma)}$.

Consider a chain $\{\varnothing\} = \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_k = S^f$ of subsets of S^f closed under inclusion and such that $|\mathcal{F}_{i+1}| = |\mathcal{F}_i| + 1$ for all i. We will define ψ, ψ', F and F' recursively on the subcomplexes $Y_{\mathcal{F}_i}$ and $\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma)$, starting with the subcomplexes $Y_{\mathcal{F}_1}$ and $\overline{\mathrm{Sal}}_{\mathcal{F}_1}$ consisting only of one 0-cell, and extending them one cell at a time. We will construct the maps in such a way that $\psi|_{Y_{\mathcal{F}_i}}$ has image contained in $\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma)$, $\psi'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma)}$ has image contained in $Y_{\mathcal{F}_i}$, and the same will hold for the homotopies F and F' for any time $t \in [0,1]$. Simultaneously we will prove by induction that, for any subset $\mathcal{F} \subseteq \mathcal{F}_i$ closed under inclusion,

- the constructed maps
$$\psi|_{Y_{\mathcal{F}_i}} \colon Y_{\mathcal{F}_i} \to \overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma) \quad \text{and} \quad \psi'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma)} \colon \overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma) \to Y_{\mathcal{F}_i},$$
restricted to $Y_{\mathcal{F}}$ and $\overline{\mathrm{Sal}}_{\mathcal{F}}(\Gamma)$, have image contained in $\overline{\mathrm{Sal}}_{\mathcal{F}}(\Gamma)$ and $Y_{\mathcal{F}}$ respectively;
- the constructed homotopies
$$F|_{Y_{\mathcal{F}_i} \times [0,1]} \colon Y_{\mathcal{F}_i} \times [0,1] \to Y_{\mathcal{F}_i} \quad \text{and} \quad F'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma) \times [0,1]} \colon \overline{\mathrm{Sal}}_{\mathcal{F}_i} \times [0,1] \to \overline{\mathrm{Sal}}_{\mathcal{F}_i},$$
restricted to $Y_{\mathcal{F}} \times [0,1]$ and $\overline{\mathrm{Sal}}_{\mathcal{F}}(\Gamma) \times [0,1]$, have image contained in $Y_{\mathcal{F}}$ and $\overline{\mathrm{Sal}}_{\mathcal{F}}(\Gamma)$ respectively.

In particular this means that the restrictions of $\psi|_{Y_{\mathcal{F}_i}}$ and $\psi'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma)}$ to $Y_{\mathcal{F}}$ and $\overline{\mathrm{Sal}}_{\mathcal{F}}(\Gamma)$ are homotopy inverses one of the other through the homotopies obtained by restricting $F|_{Y_{\mathcal{F}_i} \times [0,1]}$ and $F'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma) \times [0,1]}$ to the subspaces $Y_{\mathcal{F}} \times [0,1]$ and $\overline{\mathrm{Sal}}_{\mathcal{F}}(\Gamma) \times [0,1]$, respectively.

Define $\psi|_{Y_{\mathcal{F}_1}}$ sending the 0-cell of Y to the 0-cell of $\overline{\mathrm{Sal}}(\Gamma)$, and $\psi'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_1}(\Gamma)}$ as its inverse. Moreover define the homotopies $F|_{Y_{\mathcal{F}_1} \times [0,1]}$ and $F'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_1}(\Gamma) \times [0,1]}$ as the constant maps.

Assume now by induction to have already defined the maps $\psi|_{Y_{\mathcal{F}_i}}$, $\psi'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma)}$, $F|_{Y_{\mathcal{F}_i} \times [0,1]}$ and $F'|_{\overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma) \times [0,1]}$ for some i. To simplify the notation, set:

$$\begin{aligned} X &= Y_{\mathcal{F}_{i+1}}, & X' &= \overline{\mathrm{Sal}}_{\mathcal{F}_{i+1}}(\Gamma), \\ A &= Y_{\mathcal{F}_i}, & A' &= \overline{\mathrm{Sal}}_{\mathcal{F}_i}(\Gamma), \\ \vartheta &= \psi|_A, & \vartheta' &= \psi|_{A'}, \\ G &= F|_{A \times [0,1]}, & G' &= F'|_{A' \times [0,1]}. \end{aligned}$$

Let T be the only element of S^f which belongs to $\mathcal{F}_{i+1}$ but not to $\mathcal{F}_i$. Moreover let e^n and f^n be the n-cells corresponding to T in X and X' respectively. We want to extend ϑ to e^n, G to $e^n \times [0,1]$, ϑ' to f^n and G' to $f^n \times [0,1]$.

If $n = 1$ we simply send homeomorphically e^1 to f^1, preserving the orientation, and f^1 to e^1 with the inverse homeomorphism; the homotopies G and G' are then extended being constant on the new cells. If $n = 2$ we can apply Lemma 5.19 to observe that the boundary curve of e^2 in X is the same (via ϑ) as the boundary curve of f^2 in X'; then we extend ϑ sending e^2 to f^2 homeomorphically, preserving the boundary, and similarly we extend ϑ'. The homotopies G and G' can be extended to the new cells by Corollary 5.18.

Now we are going to deal with the case $n \geq 3$. Consider the following subsets of S^f, which are closed under inclusion:

$$\mathcal{F} = \{R \in S^f \mid R \subsetneq T\}, \quad \mathcal{F}^* = \mathcal{F} \cup \{T\}.$$

Set $\widehat{A} = Y_{\mathcal{F}}$, $\widehat{X} = Y_{\mathcal{F}^*}$, $\widehat{A}' = \overline{\mathrm{Sal}}_{\mathcal{F}}(\Gamma)$ and $\widehat{X}' = \overline{\mathrm{Sal}}_{\mathcal{F}^*}(\Gamma)$. Notice that $\mathcal{F} \subseteq \mathcal{F}_i$ because $\mathcal{F}_{i+1}$ is closed under inclusion and T is the only element in $\mathcal{F}_{i+1} \setminus \mathcal{F}_i$. Then we have the following inclusions of CW complexes.

$$\begin{tikzcd} A \arrow[r, hook] & X \\ \widehat{A} \arrow[u, hook] \arrow[r, hook] & \widehat{X} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, hook] & X' \\ \widehat{A}' \arrow[u, hook] \arrow[r, hook] & \widehat{X}' \arrow[u, hook] \end{tikzcd}$$

Let $\varphi\colon S^{n-1} \to A$ be the attaching map of the cell e^n. Notice that $\widehat{X}$ is obtained from $\widehat{A}$ by attaching the same cell e^n, so the image of φ is actually contained in $\widehat{A}$. Thus we have $\varphi\colon S^{n-1} \to \widehat{A}$. With the same argument we can deduce that the attaching map φ' of f^n has image contained in $\widehat{A}'$. Setting $\Gamma_1 = \Gamma|_T$ we have that Γ_1 is a Coxeter graph of finite type, because $T \in S^f$. Then the CW complex $\widehat{X} = Y_{\mathcal{F}^*} \simeq BA^+_{\Gamma_1}$ is a space of type $K(A_{\Gamma_1}, 1)$, by Corollary 5.11. Similarly, the CW complex $\widehat{X}' = \overline{\mathrm{Sal}}_{\mathcal{F}^*}(\Gamma) \simeq \overline{\mathrm{Sal}}(\Gamma_1)$ is also a space of type $K(A_{\Gamma_1}, 1)$ by Theorem 4.20.

By induction we know that $\vartheta\colon A \to A'$ and $\vartheta'\colon A' \to A$ are homotopy inverses one of the other through the homotopies G and G', and that the restrictions $\lambda = \vartheta|_{\widehat{A}}\colon \widehat{A} \to \widehat{A}'$ and $\lambda' = \vartheta'|_{\widehat{A}'}\colon \widehat{A}' \to \widehat{A}$ are homotopy inverses one of the other through the restricted homotopies $H = G|_{\widehat{A}\times[0,1]}\colon \widehat{A} \times [0,1] \to \widehat{A}$ and $H' = G'|_{\widehat{A}'\times[0,1]}\colon \widehat{A}' \times [0,1] \to \widehat{A}'$.

Since $\pi_{n-1}(\widehat{X}) = 0$ and $\pi_{n-1}(\widehat{X}') = 0$, the maps λ and λ' can be extended to maps

$$\tilde{\lambda}\colon \widehat{X} \to \widehat{X}', \quad \tilde{\lambda}'\colon \widehat{X}' \to \widehat{X}.$$

Extend H to the map $\bar{H}\colon \widehat{A} \times [0,1] \cup \widehat{X} \times \{0,1\} \to \widehat{X}$ as follows: $\bar{H}|_{\widehat{X}\times\{0\}} = \tilde{\lambda}' \circ \tilde{\lambda}\colon \widehat{X} \to \widehat{X}$ and $\bar{H}|_{\widehat{X}\times\{1\}} = \mathrm{id}_{\widehat{X}}$. Then attaching the $(n+1)$-cell $e^n \times [0,1]$ to $\widehat{A} \times [0,1] \cup \widehat{X} \times \{0,1\}$ yields the space $\widehat{X} \times [0,1]$. Since $\pi_n(\widehat{X}) = 0$, $\bar{H}$ can be extended to a homotopy $\tilde{H}\colon \widehat{X} \times [0,1] \to \widehat{X}$ between $\tilde{\lambda}' \circ \tilde{\lambda}$ and the identity map of $\widehat{X}$. Extend similarly H' to a homotopy $\tilde{H}'$ between $\tilde{\lambda} \circ \tilde{\lambda}'$ and the identity map of $\widehat{X}'$.

Finally extend ϑ to e^n by gluing it with $\tilde{\lambda}$ (these two maps coincide on $A \cap \widehat{X} = \widehat{A}$), extend G to $e^n \times [0,1]$ by gluing it with $\tilde{H}$ (these two homotopies coincide on $A \times [0,1] \cap \widehat{X} \times [0,1] = \widehat{A} \times [0,1]$), and do the same for ϑ' and G'. Call $\tilde{\vartheta}$, $\tilde{G}$, $\tilde{\vartheta}'$ and $\tilde{G}'$ the extended maps. By construction $\tilde{G}$ is a homotopy between $\tilde{\vartheta}' \circ \tilde{\vartheta}$ and id_X, and $\tilde{G}'$ is a homotopy between $\tilde{\vartheta} \circ \tilde{\vartheta}'$ and $\mathrm{id}_{X'}$.

To complete our induction argument we only need to prove that, for any subset $\mathcal{F} \subseteq \mathcal{F}_{i+1}$ closed under inclusion, the restrictions $\tilde{\vartheta}|_{Y_\mathcal{F}}$ and $\tilde{G}|_{Y_\mathcal{F}\times[0,1]}$ have image contained in $\overline{\mathrm{Sal}}_\mathcal{F}(\Gamma)$ and $Y_\mathcal{F}$ respectively (the analogous property for $\tilde{\vartheta}'$ and $\tilde{G}'$ will hold similarly). If $T \notin \mathcal{F}$ then $\mathcal{F} \subseteq \mathcal{F}_i$, so $\tilde{\vartheta}|_{Y_\mathcal{F}}$ and $\tilde{G}|_{Y_\mathcal{F}\times[0,1]}$ are restrictions of ϑ and G, and our claim follows by induction. So we can assume $T \in \mathcal{F}$. If we set $\mathcal{F}' = \mathcal{F} \setminus \{T\}$ then $\mathcal{F}' \subseteq \mathcal{F}_i$, so the claim holds for $\mathcal{F}'$ by induction. If e^n is the cell corresponding to T, then by construction the restrictions $\tilde{\vartheta}|_{e^n}$ and $\tilde{G}|_{e^n\times[0,1]}$ have image contained in $\overline{\mathrm{Sal}}_{\mathcal{F}^*}(\Gamma)$ and $Y_{\mathcal{F}^*}$ respectively, where $\mathcal{F}^* = \{R \in S^f \mid R \subseteq T\}$. Since $\mathcal{F}$ is closed under inclusion and $T \in \mathcal{F}$, then $\mathcal{F}^* \subseteq \mathcal{F}$. Therefore the restrictions $\tilde{\vartheta}|_{Y_\mathcal{F}}$ and $\tilde{G}|_{Y_\mathcal{F}\times[0,1]}$ have image contained in $\overline{\mathrm{Sal}}_{\mathcal{F}'}(\Gamma) \cup \overline{\mathrm{Sal}}_{\mathcal{F}^*}(\Gamma) = \overline{\mathrm{Sal}}_\mathcal{F}(\Gamma)$ and $Y_{\mathcal{F}'} \cup Y_{\mathcal{F}^*} = Y_\mathcal{F}$ respectively. $\square$

Bibliography

[Bou68] N. Bourbaki, *Éléments de mathématique: Fasc. XXXIV. Groupes et algèbres de Lie; Chap. 4, Groupes de Coxeter et systèmes de Tits; Chap. 5; Chap. 6, Systèmes de racines*, Hermann, 1968.

[Bri73] E. Brieskorn, Sur les groupes de tresses [d'après VI Arnol'd], in *Séminaire Bourbaki vol. 1971/72 Exposés 400–417*, pages 21–44, Springer, 1973.

[Bro89] K. S. Brown, *Buildings*, Springer, 1989.

[BS72] E. Brieskorn and K. Saito, *Artin-gruppen und Coxeter-gruppen*, Inventiones mathematicae **17**(4), 245–271 (1972).

[BW02] E. Batzies and V. Welker, *Discrete Morse theory for cellular resolutions*, Journal fur die Reine und Angewandte Mathematik , 147–168 (2002).

[BZ92] A. Björner and G. M. Ziegler, *Combinatorial stratification of complex arrangements*, Journal of the American Mathematical Society **5**(1), 105–149 (1992).

[CD95] R. Charney and M. W. Davis, *The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups*, Journal of the American Mathematical Society , 597–627 (1995).

[Cha] R. Charney, *Problems related to Artin groups*, `http://people.brandeis.edu/~charney/papers/Artin_probs.pdf`.

[Cha00] M. K. Chari, *On discrete Morse functions and combinatorial decompositions*, Discrete Mathematics **217**(1), 101–113 (2000).

[CMS08a] F. Callegaro, D. Moroni and M. Salvetti, *Cohomology of affine Artin groups and applications*, Transactions of the American Mathematical Society **360**(8), 4169–4188 (2008).

[CMS08b] F. Callegaro, D. Moroni and M. Salvetti, *Cohomology of Artin groups of type $\tilde{A}_n$, B_n and applications*, Geometry & Topology Monographs **13**, 85–104 (2008).

[CMS10] F. Callegaro, D. Moroni and M. Salvetti, *The $K(\pi, 1)$ problem for the affine Artin group of type B_n and its cohomology*, J. Eur. Math. Soc.(JEMS) **12**(1), 1–22 (2010).

[Cox34] H. S. Coxeter, *Discrete groups generated by reflections*, Annals of Mathematics , 588–621 (1934).

[Dav08] M. Davis, *The geometry and topology of Coxeter groups*, volume 32, Princeton University Press, 2008.

[DCS96] C. De Concini and M. Salvetti, *Cohomology of Artin groups*, Math. Res. Lett **3**(2), 293–297 (1996).

[Del72] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Inventiones mathematicae **17**(4), 273–302 (1972).

[Dob06] N. E. Dobrinskaya, *Configuration spaces of labeled particles and finite Eilenberg-MacLane complexes*, Proceedings of the Steklov Institute of Mathematics **252**(1), 30–46 (2006).

[For98] R. Forman, *Morse theory for cell complexes*, Advances in mathematics **134**(1), 90–145 (1998).

[Gar69] F. A. Garside, *The braid group and other groups*, The Quarterly Journal of Mathematics **20**(1), 235–254 (1969).

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[Hen85] H. Hendriks, *Hyperplane complements of large type*, Inventiones mathematicae **79**(2), 375–381 (1985).

[Hum92] J. E. Humphreys, *Reflection groups and Coxeter groups*, volume 29, Cambridge university press, 1992.

[KDT08] C. Kassel, O. Dodane and V. Turaev, *Braid groups*, Springer Science & Business Media, 2008.

[Mil57] J. Milnor, *The geometric realization of a semi-simplicial complex*, Annals of Mathematics , 357–362 (1957).

[MS76] D. McDuff and G. Segal, *Homology fibrations and the "group-completion" theorem*, Inventiones mathematicae **31**(3), 279–284 (1976).

[Oko79] C. Okonek, *Das $K(\pi$, 1)-Problem für die affinen Wurzelsysteme vom Typ A_n, C_n*, Mathematische Zeitschrift **168**(2), 143–148 (1979).

[Ozo13] V. Ozornova, *Discrete Morse theory and a reformulation of the $K(\pi, 1)$-conjecture*, arXiv preprint arXiv:1309.1337 (2013).

[Par02] L. Paris, *Artin monoids inject in their groups*, Commentarii Mathematici Helvetici **77**(3), 609–637 (2002).

[Par12] L. Paris, $K(\pi, 1)$ *conjecture for Artin groups*, arXiv preprint arXiv:1211.7339 (2012).

[Sal87] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^N$, Inventiones mathematicae **88**(3), 603–618 (1987).

[Sal93] M. Salvetti, *On the homotopy theory of complexes associated to metrical-hemisphere complexes*, Discrete Mathematics **113**(1), 155–177 (1993).

[Sal94] M. Salvetti, *The homotopy type of Artin groups*, Math. Res. Lett **1**(5), 565–577 (1994).

[Seg73] G. Segal, *Configuration-spaces and iterated loop-spaces*, Inventiones Mathematicae **21**(3), 213–221 (1973).

[Tit66] J. Tits, *Normalisateurs de tores I. Groupes de Coxeter étendus*, Journal of algebra **4**(1), 96–116 (1966).

[Tit69] J. Tits, Le probleme des mots dans les groupes de Coxeter, in *Symposia Mathematica*, volume 1, pages 175–185, 1969.

[vdL83] H. van der Lek, *The homotopy type of complex hyperplane complements*, PhD thesis, Katholieke Universiteit te Nijmegen, 1983.

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