



SCUOLA NORMALE SUPERIORE

TOPOLOGY AND COMBINATORICS
OF AFFINE REFLECTION ARRANGEMENTS

Tesi di Perfezionamento in Matematica

Candidato

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Anno accademico 2018/2019

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Contents

Introduction	9
I Background	13
1 Combinatorial tools	15
1.1 Posets	15
1.2 Discrete Morse theory	17
1.3 Algebraic discrete Morse theory	18
1.4 Shellability	19
1.5 Lexicographic shellability	21
2 Hyperplane arrangements, Coxeter groups, Artin groups	23
2.1 Hyperplane arrangements	23
2.2 Posets associated to real arrangements	24
2.3 Topology of the complement	25
2.4 Coxeter groups	26
2.5 Root systems	28
2.6 Artin groups	29
2.7 Abelian arrangements	31
II Minimality	37
3 Acyclic matchings on the Salvetti complex	39
3.1 Introduction	39
3.2 Decomposition of the Salvetti complex	40
3.3 Acyclic matchings from valid orders	45
3.4 Remarks on the poset of faces	52
4 Euclidean matchings and minimality	55
4.1 Introduction	55
4.2 Euclidean matchings	56
4.3 Minimality	60

4.4	Brieskorn's Lemma and naturality	63
4.5	Local homology of line arrangements	66
III Local homology		73
5	Precise matchings on weighted sheaves	75
5.1	Introduction	75
5.2	Weighted sheaves over posets	76
5.3	Decomposition and filtration	77
5.4	Discrete Morse theory for weighted complexes	78
5.5	Precise matchings	80
6	Local homology of Artin groups	87
6.1	Introduction	87
6.2	Weighted sheaves for Artin groups	88
6.3	Braid groups and independence complexes	90
6.4	Precise matchings for Artin groups	95
6.5	Reduction to the irreducible cases	98
6.6	Weight of irreducible components	100
6.7	Case A_n	101
6.8	Case B_n	105
6.9	Case D_n	113
6.10	Case \tilde{A}_n	116
6.11	Case \tilde{B}_n	121
6.12	Case \tilde{C}_n	122
6.13	Case \tilde{D}_n	130
6.14	Exceptional cases	132
6.15	Local homology of low-dimensional groups	134
IV Shellability		141
7	Shellability of arrangements defined by root systems	143
7.1	Introduction	143
7.2	Labeled partitions	144
7.3	Posets of layers	146
7.4	EL-shellability of posets of labeled partitions	146
7.5	EL-shellability of posets of layers	152
7.6	Homotopy type	153
8	Shellability of generalized Dowling posets	161
8.1	Introduction	161
8.2	Generalized Dowling posets	162
8.3	EL-shellability of generalized Dowling posets	163

8.4 Homotopy type	168
8.5 Shellability of subposets	173
V Classifying spaces	179
9 Finite classifying spaces for affine Artin groups	181
9.1 Introduction	181
9.2 Interval groups	182
9.3 Dual Artin groups and braided crystallographic groups	184
9.4 Dual Salvetti complex	187
9.5 Finite classifying spaces for braided crystallographic groups .	188
9.6 Horizontal and diagonal groups	193
9.7 Finite classifying spaces for affine Artin groups	194
9.8 The $K(\pi, 1)$ conjecture for the Artin group of type \tilde{D}_4	197
Bibliography	201
Acknowledgements	211

Introduction

A fundamental impulse in the theory of hyperplane arrangements, since its very beginning, has been given by the study of reflection arrangements. The works of Fadell, Fox, and Neuwirth [FN62a, FN62b], and later of Arnold [Arn69], started the topological study of the braid arrangement, the most important example of a reflection arrangement. Among other things, they proved that the complement of the complexified braid arrangement is a classifying space — or $K(\pi, 1)$ — for the corresponding braid group. The subsequent contributions of Tits, Brieskorn, and Saito [Tit66, Tit69, BS72, Bri73] widened the field to more general reflection arrangements, and led to the introduction of Artin groups (the braid group generalizes to Artin groups in the same way as the symmetric groups generalizes to Coxeter groups). The $K(\pi, 1)$ property was proved by Deligne for any finite simplicial arrangement, thus including the case of finite reflection arrangements [Del72]. Artin groups however, and especially those corresponding to infinite reflection arrangements, still remain quite mysterious and evasive.

A particular instance of infinite reflection arrangements is given by affine (or Euclidean) reflection arrangements. The corresponding Coxeter and Artin groups are said to be *of affine type*, or simply *affine*. Some light on the nature of affine Artin groups was shed by recent works of McCammond and others [McC15, MS17].

A fundamental tool in the combinatorial study of real hyperplane arrangements is a cell complex introduced by Salvetti [Sal87, Sal94]. This complex has the homotopy type of the complement of the complexified arrangement, and is known as the *Salvetti complex*. It allowed, among other things, explicit proofs of the minimality of the complement of a finite complexified arrangement, and a systematic computation of (local) homology and cohomology of hyperplane arrangements and Artin groups [DCS96, DCPSS99, DCPS01, CS04, Cal05, Cal06, SS07, Del08, CMS08a, CMS08b, CMS10, GS09, SV13].

A recent branch in the theory of hyperplane arrangements is given by *toric arrangements* [DCP10, DCPV10], and by the more general *abelian arrangements* [Bib16]. Toric arrangements are intimately related to affine hyperplane arrangements, since a toric arrangement can be viewed as a quotient of a periodic hyperplane arrangement. A great deal of efforts is dedicated to un-

derstanding which properties of hyperplane arrangements can be translated into properties of toric and abelian arrangements, and how.



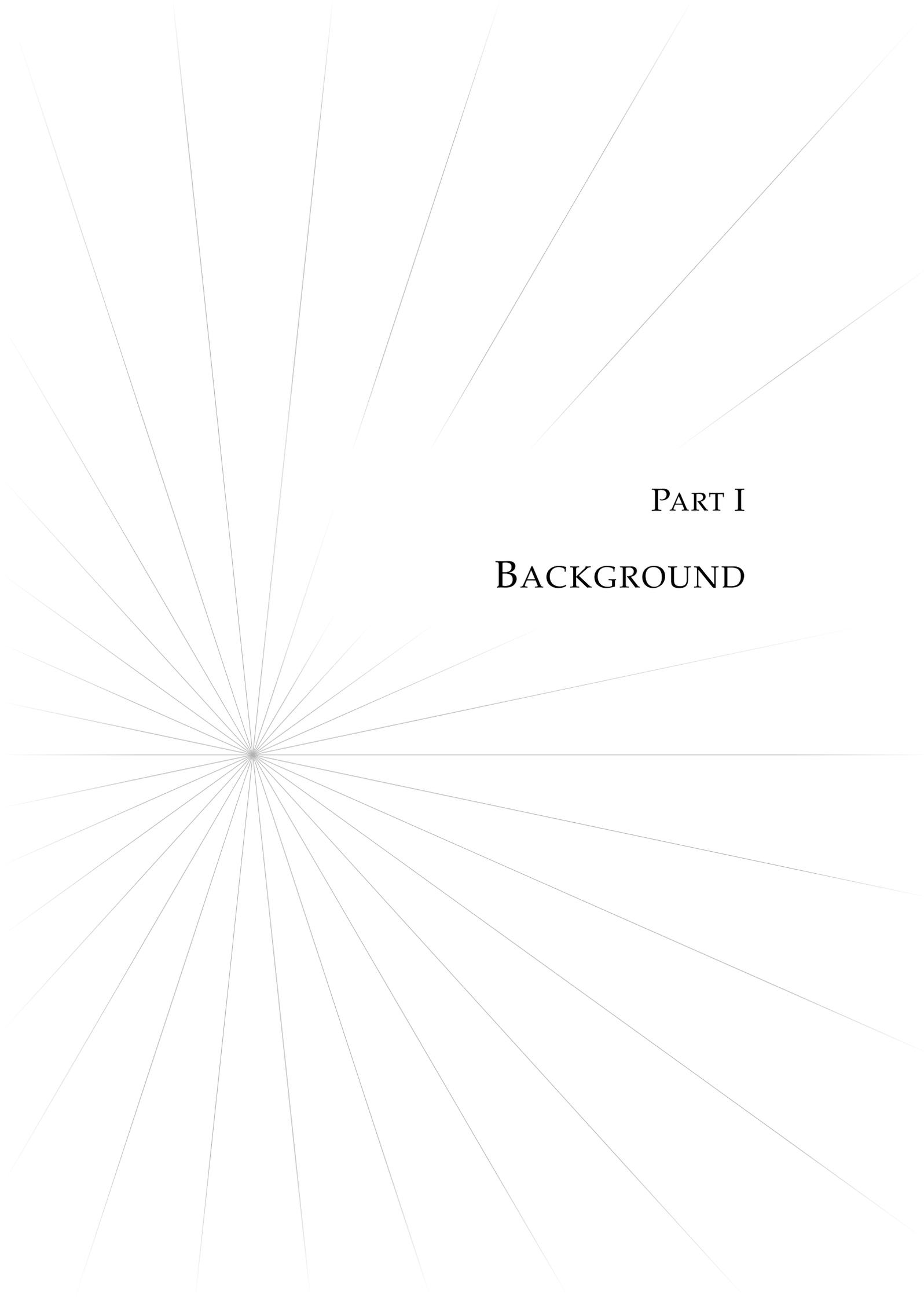
This thesis focuses on affine hyperplane arrangements, and in particular on affine reflection arrangements. Such arrangements are studied from different points of view, all sharing a deep interplay between topology and combinatorics. For this reason, a central role is played by tools of combinatorial topology, such as cell complexes [Hat02], discrete Morse theory [For98, For02], and shellability [Bjö80, BW83]. These methods are recalled in Part I, together with basic notions about hyperplane arrangements, Coxeter groups, and Artin groups.

Part II is based on joint work with Davide Lofano [LP18]. For a general locally finite affine arrangement, we describe a method to construct a minimal model for the complement of the complexification. This model is obtained from the Salvetti complex by applying discrete Morse theory. Our construction gives interesting insights also in the well-studied case of finite arrangements, for example providing a new geometric way to compute the Betti numbers of the complement. The motivation for this work was to generalize ideas of Delucchi [Del08] to non-central, and possibly infinite, arrangements.

Part III is based on two works [PS18, Pao19], the first one being in collaboration with Mario Salvetti. Here we develop a combinatorial method to study the local homology of Artin groups, which turns out to be particularly effective for groups of spherical and affine type. We systematically apply this method to all spherical and affine families. In particular, we extend to affine groups a result about torsion-freeness which was previously known for groups of spherical type (thanks to geometric considerations). We also explicitly compute the local homology for some of these families, finding new results (for groups of type \tilde{C}_n) and correcting minor errors in the literature.

Part IV is based on two works [DGP17, Pao18b], the first one being in collaboration with Emanuele Delucchi and Noriane Girard. Here we initiate the study of shellability properties of the poset of layers of abelian arrangements. For hyperplane arrangements, the poset of layers (also known as the poset of flats) is a geometric semilattice and is known to be shellable [WW85, Zie92]. This strong combinatorial condition does not transfer easily to toric arrangements (or abelian arrangements in general). We focus on the case of arrangements defined by root systems, which can be regarded as a toric/abelian analog of reflection arrangements. Our class of shellings actually applies to a larger family of posets, called generalized Dowling posets, which were recently introduced by Bibby and Gadish in the context of orbit configuration spaces [BG18].

In Part [V](#) we construct finite classifying spaces for all affine Artin groups. This improves a recent result of McCammond and Sulway [\[MS17\]](#), who proved that every affine Artin group admits a finite-dimensional classifying space. The classifying spaces we construct are very explicit, and we are able to relate them to the complement of the associated reflection arrangements. In particular, we prove the $K(\pi, 1)$ conjecture for the affine Artin group of type \tilde{D}_4 .



PART I

BACKGROUND

Combinatorial tools

1.1 POSETS

We begin this introductory chapter by recalling some basic definitions and facts about partially ordered sets (*posets*). We refer to [Sta86, Wac06] for a more detailed introduction to the subject.

Let (P, \leq) be a poset. If $p < q$ in P and there is no element $r \in P$ with $p < r < q$, then we say that q *covers* p , and write $p \triangleleft q$. Given an element $q \in P$, define $P_{\leq q} = \{p \in P \mid p \leq q\}$.

We say that P is *bounded from below* (resp. *from above*) if it contains a unique minimal (resp. maximal) element, which is usually denoted by $\hat{0}$ (resp. $\hat{1}$). The unique minimal element is also called the *bottom* element, and the unique maximal element is also called the *top* element. We say that P is *bounded* if it is bounded both from below and from above. We define the *bounded extension* \hat{P} as $P \sqcup \{\hat{0}, \hat{1}\}$, where new elements $\hat{0}$ and $\hat{1}$ are added (even if P already has a bottom or top element). If P is bounded from below, then every element q such that $\hat{0} \triangleleft q$ is called an *atom*.

A (finite) *chain* in P is a totally ordered sequence $p_0 < p_1 < \dots < p_n$ of elements of P . A chain of $n + 1$ elements is conventionally said to have *length* n .

If $p \leq q$, the *interval* $[p, q]$ in P is the set of all elements $r \in P$ such that $p \leq r \leq q$. We say that P is *ranked* if, for every $p \leq q$, all the maximal chains in $[p, q]$ have the same (finite) length. Then there exists a *rank function* $\text{rk}: P \rightarrow \mathbb{N}$ such that $\text{rk}(q) - \text{rk}(p)$ is the length of any maximal chain in $[p, q]$. The rank of the poset is defined as the maximal length of a chain (and can be infinite). In the following chapters, we will usually deal with ranked posets having a finite rank.

A poset P is said to be a *meet semilattice* if every pair of elements $p, q \in P$ has a unique maximal common lower bound $p \wedge q$ (which is called *meet*). Similarly, P is a *join semilattice* if every pair of elements $p, q \in P$ has a unique

minimal common upper bound $p \vee q$ (called *join*). If P is both a meet and a join semilattice, then it is said to be a *lattice*.

In the theory of hyperplane arrangements, an important class of posets is given by geometric lattices and semilattices. A *geometric lattice* is a finite lattice P that is semimodular (for all $p, q \in P$, the join $p \vee q$ covers q whenever p covers the meet $p \wedge q$) and atomic (every element of P is a join of atoms). Every geometric lattice is ranked. A ranked meet semilattice P is a *geometric semilattice* if

- (i) every interval is a geometric lattice;
- (ii) for every element $p \in P$ and for every subset A of atoms whose join exists, if $\text{rk}(p) < \text{rk}(\bigvee A)$, then there exists an atom $a \in A$ such that $a \not\leq p$ and $a \vee p$ exists.

Given a poset P , its *Hasse diagram* is the graph with vertex set P and having an edge (p, q) whenever $p \lessdot q$. An edge (p, q) is sometimes indicated as $q \rightarrow p$.

The *order complex* $\Delta(P)$ is the simplicial complex with vertex set P and with faces given by chains of P . One often says that P has some topological property (e.g. it is contractible) if its order complex has that property. Notice that, if P is bounded from below or from above, then $\Delta(P)$ is a cone and hence it is contractible.

If P is a poset such that every interval is finite, we can define its *Möbius function* as follows. Denote by $\mathcal{I}(P)$ the set of all intervals of P , which can be also described as the set of pairs $(p, q) \in P \times P$ such that $p \leq q$. Then the Möbius function $\mu_P: \mathcal{I}(P) \rightarrow \mathbb{Z}$ is recursively defined by these relations:

$$\begin{aligned} \mu_P(p, p) &= 1 && \text{for all } p \\ \mu_P(p, q) &= - \sum_{p \leq r < q} \mu_P(p, r) && \text{for all } p < q. \end{aligned}$$

If P is also ranked and bounded from below, then we can define its *characteristic polynomial*

$$\chi_P(t) = \sum_{p \in P} \mu_P(\hat{0}, p) t^{\text{rk } p}.$$

The following theorem relates the Möbius function with the reduced Euler characteristic of the order complex.

Theorem 1.1.1 (Hall [Sta86, Theorem 3.8.6]). Let P be a finite poset, and let $\hat{P} = P \sqcup \{\hat{0}, \hat{1}\}$ be its bounded extension. Then

$$\mu_P(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P)).$$

1.2 DISCRETE MORSE THEORY

In this section we recall the main concepts of Forman's discrete Morse theory [For98, For02]. We follow the point of view of Chari [Cha00], using acyclic matchings instead of discrete Morse functions, and we make use of the generality of [Bat02, Section 3] for the case of infinite CW complexes.

Let (P, \leq) be a ranked poset. Denote by G the Hasse diagram of P , and by $\mathcal{E} = \mathcal{E}(P) = \{(p, q) \in P \times P \mid q \lessdot p\}$ the set of edges of G .

Given a subset \mathcal{M} of \mathcal{E} , we can orient all edges of G in the following way: an edge $(p, q) \in \mathcal{E}$ is oriented from p to q if the pair does not belong to \mathcal{M} , otherwise in the opposite direction. Denote this oriented graph by $G_{\mathcal{M}}$.

Definition 1.2.1 (Acyclic matching [Cha00]). A *matching* on P is a subset $\mathcal{M} \subseteq \mathcal{E}$ such that every element of P appears in at most one edge of \mathcal{M} . A matching \mathcal{M} is *acyclic* if the graph $G_{\mathcal{M}}$ has no directed cycle.

Given a matching \mathcal{M} on P , an *alternating path* is a directed path in $G_{\mathcal{M}}$ such that two consecutive edges of the path do not belong both to \mathcal{M} or both to $\mathcal{E} \setminus \mathcal{M}$. The elements of P that do not appear in any edge of \mathcal{M} are called *critical* (with respect to the matching \mathcal{M}).

Definition 1.2.2 (Grading [Bat02]). Let Q be a poset. An order-preserving map $\varphi: P \rightarrow Q$ is called a *Q-grading* of P . The Q -grading φ is *compact* if $\varphi^{-1}(Q_{\leq q}) \subseteq P$ is finite for all $q \in Q$. A matching \mathcal{M} on P is *homogeneous* with respect to the Q -grading φ if $\varphi(p) = \varphi(p')$ for all $(p, p') \in \mathcal{M}$. An acyclic matching \mathcal{M} is *proper* if it is homogeneous with respect to some compact grading.

The following is a direct consequence of the definition of a proper matching (cf. [Bat02, Definition 3.2.5 and Remark 3.2.17]).

Lemma 1.2.3 ([Bat02]). Let \mathcal{M} be a proper acyclic matching on a poset P , and let $p \in P$. Then there is a finite number of alternating paths starting from p , and each of them has a finite length.

We are ready to state the main theorem of discrete Morse theory. This particular formulation follows from [Bat02, Theorem 3.2.14 and Remark 3.2.17].

Theorem 1.2.4 ([For98, Cha00, Bat02]). Let X be a regular CW complex, and let P be its poset of cells. If \mathcal{M} is a proper acyclic matching on P , then X is homotopy equivalent to a CW complex $X^{\mathcal{M}}$ (called the *Morse complex* of \mathcal{M}) with cells in dimension-preserving bijection with the critical cells of X .

The construction of the Morse complex is explicit in terms of the CW complex X and the matching \mathcal{M} (see for example [Bat02]). This allows to obtain relations between the incidence numbers with \mathbb{Z} coefficients in the Morse complex and incidence numbers $[\sigma : \tau]$ of the starting complex.

Theorem 1.2.5 ([Bat02, Theorem 3.4.2]). Let X be a regular CW complex, P its poset of cells and \mathcal{M} a proper acyclic matching on P . Let $X^{\mathcal{M}}$ be the Morse complex of \mathcal{M} . Given two critical cells $\sigma, \tau \in X$ with $\dim \sigma = \dim \tau + 1$, denote by $\sigma_{\mathcal{M}}$ and $\tau_{\mathcal{M}}$ the corresponding cells in $X^{\mathcal{M}}$. Then the incidence number between $\sigma_{\mathcal{M}}$ and $\tau_{\mathcal{M}}$ in $X_{\mathcal{M}}$ is given by

$$[\sigma_{\mathcal{M}} : \tau_{\mathcal{M}}]^{\mathcal{M}} = \sum_{\gamma \in \Gamma(\sigma, \tau)} m(\gamma),$$

where $\Gamma(\sigma, \tau)$ is the set of all alternating paths between σ and τ . If $\gamma \in \Gamma(\sigma, \tau)$ is of the form

$$\sigma = \sigma_0 \succ \tau_1 \prec \sigma_1 \succ \cdots \prec \sigma_k \succ \tau,$$

then $m(\gamma)$ is given by

$$m(\gamma) = (-1)^k [\sigma_k : \tau] \prod_{i=1}^k [\sigma_{i-1} : \tau_i] [\sigma_i : \tau_i].$$

The following is a standard tool to construct acyclic matchings.

Theorem 1.2.6 (Patchwork theorem [Koz07, Theorem 11.10]). Let $\varphi: P \rightarrow Q$ be a Q -grading of P . For all $q \in Q$, assume to have an acyclic matching $\mathcal{M}_q \subseteq \mathcal{E}$ that involves only elements of the subposet $\varphi^{-1}(q) \subseteq P$. Then the union of these matchings is itself an acyclic matching on P .

Remark 1.2.7. The problem of computing optimal acyclic matchings is well-studied, and known to be algorithmically hard [EG96, JP06, MF08, BL14, BLPS16, Tan16, Pao18a]. However, there are algorithms which work sufficiently well in practical instances [JP06, BL14]. Computational aspects played a role in our work, especially in the computation of the local homology of exceptional Artin groups (in Chapter 6), and in the proof of the $K(\pi, 1)$ conjecture for the Artin group of type \tilde{D}_4 (in Chapter 9).

1.3 ALGEBRAIC DISCRETE MORSE THEORY

A purely algebraic version of discrete Morse theory allows to simplify free chain complexes which do not necessarily arise as cellular chain complexes [JW05, Koz05, Skö06].

Let R be a commutative ring with unity. Consider a chain complex of free R -modules

$$C_* = \cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots,$$

with a fixed R -basis P_n of each module C_n . Given $\sigma \in P_n$ and $\tau \in P_{n-1}$, denote by $[\sigma : \tau] \in R$ the coefficient of τ in the expansion of $\partial_n(\sigma)$ in terms of the basis P_{n-1} . In analogy with the topological version, we call this the incidence

number between σ and τ . Set $[\sigma : \tau] = 0$ if the dimensions do not match, i.e. if $\dim \sigma \neq \dim \tau + 1$. The union $P = \bigcup_{n \in \mathbb{Z}} P_n$ of the bases can be regarded as a ranked poset, with cover relations given by $\sigma \succ \tau$ if $[\sigma : \tau] \neq 0$.

Theorem 1.3.1 ([JW05, Koz05, Skö06]). Let (C_*, P) be a free chain complex with a basis, and let \mathcal{M} be a proper acyclic matching on P such that $[\sigma : \tau]$ is invertible in R whenever $(\tau, \sigma) \in \mathcal{M}$. Then C_* is chain-homotopy equivalent to a free chain complex $C_*^{\mathcal{M}}$, called the *algebraic Morse complex*, which is defined as follows:

- the R -module $C_n^{\mathcal{M}}$ is freely generated by a basis $P_n^{\mathcal{M}}$, having one element $\sigma_{\mathcal{M}}$ for every critical element σ of P_n ;
- the boundary of a generator $\sigma_{\mathcal{M}} \in P_n^{\mathcal{M}}$ is given by

$$\partial_n^{\mathcal{M}}(\sigma_{\mathcal{M}}) = \sum_{\tau_{\mathcal{M}} \in P_{n-1}^{\mathcal{M}}} [\sigma_{\mathcal{M}} : \tau_{\mathcal{M}}]^{\mathcal{M}} \tau_{\mathcal{M}},$$

where the incidence number is a sum $[\sigma_{\mathcal{M}} : \tau_{\mathcal{M}}]^{\mathcal{M}} = \sum_{\gamma \in \Gamma(\sigma, \tau)} m(\gamma)$ of contributions of the form

$$m(\gamma) = (-1)^k [\sigma_k : \tau] \prod_{i=1}^k [\sigma_{i-1} : \tau_i]^{-1} [\sigma_i : \tau_i].$$

Notice that the boundary formula of Theorem 1.3.1 is the same as that of Theorem 1.2.5, except for an inverse. However, an invertible incidence number $[\sigma_{i-1} : \tau_i]$ in the hypotheses of Theorem 1.2.5 must be equal to ± 1 , and therefore $[\sigma_{i-1} : \tau_i]^{-1} = [\sigma_{i-1} : \tau_i]$.

A version of discrete Morse theory for non-free chain complexes was introduced in [SV13], and is further developed in Part III, in order to study the local homology of Artin groups.

1.4 SHELLABILITY

In the last part of this chapter, we are going to recall the concept of shellability. For a more detailed exposition on the subject, see [BW83, BW96, BW97, Wac06, Zie12].

Recall that a regular CW complex is *pure* of dimension d if all its maximal cells have dimension d .

Definition 1.4.1 ([BW97, Definition 13.1]). Let X be a finite regular CW complex. For each $\sigma \in X$, denote by $\partial\sigma$ the subcomplex consisting of all proper faces of σ . A linear ordering $\sigma_1, \sigma_2, \dots, \sigma_t$ of the maximal cells of X is called a *shelling* if either $\dim X = 0$, or $\dim X \geq 1$ and the following conditions are satisfied:

- (i) $\partial\sigma_1$ admits a shelling;
- (ii) $\partial\sigma_j \cap \left(\bigcup_{i=1}^{j-1} \partial\sigma_i \right)$ is pure of dimension $(\dim \sigma_j - 1)$, for $2 \leq j \leq t$;
- (iii) $\partial\sigma_j$ admits a shelling in which the $(\dim \sigma_j - 1)$ -cells of $\partial\sigma_j \cap \left(\bigcup_{i=1}^{j-1} \partial\sigma_i \right)$ come first, for $2 \leq j \leq t$.

A complex that admits a shelling is said to be *shellable*.

Shellability has many important consequences. One of the most notable is the following.

Theorem 1.4.2 ([BW97, Corollary 13.3]). If X is a shellable complex, then it has the homotopy type of a wedge of spheres. If X is a shellable complex which is pure of dimension d , then it has the homotopy type of a wedge of d -dimensional spheres.

Following [Zie12], a *polyhedron* is a finite intersection of closed halfspaces in \mathbb{R}^n , and a *polytope* is a bounded polyhedron. A *facet* of a polytope is a face of codimension 1. The first important instance of shellable complexes was given by the boundary complex of a polytope. In fact, the boundary complex of a polytope admits many shellings, as described by the following two results.

Theorem 1.4.3 ([BM72], [Zie12, Theorem 8.12]). Let $P \subseteq \mathbb{R}^n$ be a n -dimensional polytope, and let $x \in \mathbb{R}^n$ be a point outside P . If x lies in general position (that is, not in the affine hull of a facet of P), then the boundary complex ∂P has a shelling in which the facets of P that are visible from x come first.

Lemma 1.4.4 ([Zie12, Lemma 8.10]). If F_1, F_2, \dots, F_s is a shelling order for the boundary of a polytope P , then so is the reverse order F_s, F_{s-1}, \dots, F_1 .

When the given complex is the order complex of a poset, the following theorem gives an interpretation of the number of spheres in terms of the characteristic polynomial of the poset. It is a direct consequence of Hall's theorem (Theorem 1.1.1) and of the definition of the characteristic polynomial.

Theorem 1.4.5 (cf. [DGP17, Theorem 2]). Let P be a ranked poset, and suppose that $\Delta(P)$ is shellable. Then $\Delta(P)$ is a wedge of $(-1)^d \chi_{\hat{P}}(0)$ spheres of dimension $\text{rk}(P)$.

Notice that $\chi_{\hat{P}}(0) = \mu_{\hat{P}}(\hat{0}, \hat{1}) = -\chi_{P'}(1)$, where $P' = P \cup \{\hat{0}\}$.

1.5 LEXICOGRAPHIC SHELLABILITY

There are many techniques to construct shellings of order complexes of posets. One of these is *lexicographic shellability* [BW83, BW96], which we recall in this section.

Let P be a bounded poset. As in Section 1.2, denote by $\mathcal{E}(P)$ the set of edges of the Hasse diagram of P . An *edge labeling* of P is a map $\lambda: \mathcal{E}(P) \rightarrow \Lambda$, where Λ is some poset. Given an edge labeling λ , each maximal chain $c = (x < z_1 < \cdots < z_t < y)$ between any two elements $x \leq y$ has an associated word

$$\lambda(c) = \lambda(x, z_1) \lambda(z_1, z_2) \cdots \lambda(z_t, y).$$

We say that the chain c is *increasing* if the associated word $\lambda(c)$ is strictly increasing, and *decreasing* if the associated word is weakly decreasing. Maximal chains in a fixed interval $[x, y] \subseteq P$ can be compared lexicographically (i.e. by using the lexicographic order on the corresponding words).

Definition 1.5.1. Let P be a bounded poset. An *edge-lexicographical labeling* (or simply *EL-labeling*) of P is an edge labeling such that in each closed interval $[x, y] \subseteq P$ there is a unique increasing maximal chain, and this chain lexicographically precedes all other maximal chains of $[x, y]$.

The main motivation for introducing EL-labelings of posets is given by the following theorem.

Theorem 1.5.2 ([BW96, Theorem 5.8]). Let P be a bounded poset with an EL-labeling. Then the lexicographic order of the maximal chains of P is a shelling of the order complex $\Delta(P)$. In addition, the corresponding order of the maximal chains of $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$ is a shelling of $\Delta(\bar{P})$.

The shelling induced by an EL-labeling is called an *EL-shelling*. A bounded poset that admits an EL-labeling is said to be *EL-shellable*. If P is an EL-shellable poset, then $\Delta(\bar{P})$ is homotopy equivalent to a wedge of spheres indexed by the decreasing maximal chains of P . More precisely, if D_P is the set of decreasing maximal chains of P , then

$$\Delta(\bar{P}) \simeq \bigvee_{c \in D_P} S^{|c|-3}.$$

Important examples of EL-shellable posets are given by geometric lattices [Sta74, Bjö80], and more generally by geometric semilattices [WW85, Zie92].

Let P_1 and P_2 be bounded posets that admit EL-labelings $\lambda_1: \mathcal{E}(P_1) \rightarrow \Lambda_1$ and $\lambda_2: \mathcal{E}(P_2) \rightarrow \Lambda_2$, respectively. Assume that Λ_1 and Λ_2 are disjoint and totally ordered. Let $\lambda: \mathcal{E}(P_1 \times P_2) \rightarrow \Lambda_1 \cup \Lambda_2$ be the edge labeling of $P_1 \times P_2$ defined as follows:

$$\begin{aligned} \lambda((a, b), (c, b)) &= \lambda_1(a, c) \\ \lambda((a, b), (a, d)) &= \lambda_2(b, d). \end{aligned}$$

Theorem 1.5.3 ([BW97, Proposition 10.15]). Fix any shuffle of the total orders on Λ_1 and Λ_2 , to get a total order on $\Lambda_1 \cup \Lambda_2$. Then the product edge labeling λ defined above is an EL-labeling.

When P is a shellable bounded poset, certain subposets $Q \subseteq P$ are also shellable via an induced labeling. A general criterion is given by [BW97, Theorem 10.2]. The following is a particular case.

Lemma 1.5.4 ([DGP17, Lemma 1]). Let P be a bounded and ranked poset, with an EL-labeling λ . Let $Q \subseteq P$ be a ranked subposet of P containing $\hat{0}$ and $\hat{1}$, with rank function given by the restriction of the rank function of P . Suppose that, for all $x \leq y$ in Q , the unique increasing maximal chain in $[x, y] \subseteq P$ is also contained in Q . Then the edge labeling $\lambda|_{\mathcal{E}(Q)}$ is an EL-labeling of Q .

Hyperplane arrangements, Coxeter groups, Artin groups

2.1 HYPERPLANE ARRANGEMENTS

We begin this chapter by reviewing basic notions and results about hyperplane arrangements. See [OT13] for a general reference.

A *hyperplane arrangement* is a set of affine hyperplanes in some finite-dimensional vector space \mathbb{K}^n . We will be interested in the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, so that the vector space \mathbb{K}^n is endowed with the usual Euclidean topology. Hyperplane arrangements are assumed to be locally finite (every compact subset of \mathbb{K}^n intersects only a finite number of hyperplanes), unless explicitly stated.

Given a hyperplane arrangement \mathcal{A} in \mathbb{R}^n , we define its *complexification* $\mathcal{A}_{\mathbb{C}}$ to be the hyperplane arrangement in \mathbb{C}^n consisting of the complexified hyperplanes $H_{\mathbb{C}} = H \otimes_{\mathbb{R}} \mathbb{C}$ for $H \in \mathcal{A}$.

An arrangement is *central* if all its hyperplanes pass through a common point. This point is usually taken to be the origin, so that the hyperplanes have linear equations. Notice that a locally finite central arrangement is necessarily finite.

An arrangement in \mathbb{K}^n is *essential* if the \mathbb{K} -vector space spanned by the normals to its hyperplanes has dimension n .

Example 2.1.1 (Braid arrangement). Consider the central arrangement \mathcal{A} in \mathbb{R}^n consisting of the hyperplanes $H_{ij} = \{x_i = x_j\}$ for $1 \leq i < j \leq n$. This arrangement is called the *braid arrangement*, or the reflection arrangement of type A_{n-1} (see Section 2.4). Since it is not essential, one usually restricts it to $\mathbb{R}^{n-1} = \{x_1 + \cdots + x_n = 0\} \subseteq \mathbb{R}^n$, as in Figure 2.1.

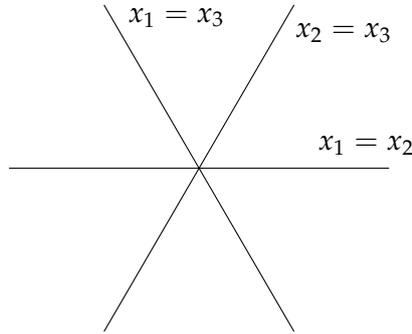


Figure 2.1: The braid arrangement of type A_2 in the plane $\{x_1 + x_2 + x_3 = 0\} \subseteq \mathbb{R}^3$.

2.2 POSETS ASSOCIATED TO REAL ARRANGEMENTS

Let \mathcal{A} be a hyperplane arrangement in \mathbb{R}^n . In this section we are going to introduce some combinatorial objects associated to \mathcal{A} .

The arrangement \mathcal{A} gives rise to a stratification of \mathbb{R}^n into subspaces called *faces* (see [Bou68, Chapter 5]). It is more convenient for us to work with the closure of these subspaces, so we assume from now on that the faces are closed. By *relative interior* of a face F we mean the topological interior of F inside the affine span of F . The faces of codimension 0 are called *chambers*. Denote the set of faces by $\mathcal{F} = \mathcal{F}(\mathcal{A})$, and the set of the chambers by $\mathcal{C} = \mathcal{C}(\mathcal{A})$. The set \mathcal{F} has a natural partial order: $F \preceq G$ if and only if $F \supseteq G$. The poset \mathcal{F} is called the *face poset* of \mathcal{A} , and it is ranked by codimension.

Given two chambers $C, C' \in \mathcal{C}$, let $s(C, C') \subseteq \mathcal{A}$ be the set of hyperplanes which separate C and C' . Also, denote by $\mathcal{W}_C \subseteq \mathcal{A}$ the set of hyperplanes that intersect C in a face of codimension 1. These hyperplanes are called the *walls* of C .

There is a second important poset associated to a hyperplane arrangement \mathcal{A} , namely the *poset of flats* (or *poset of intersections*), denoted by $\mathcal{L} = \mathcal{L}(\mathcal{A})$. It consists of all possible intersections of hyperplanes in \mathcal{A} . An element $X \in \mathcal{L}$ is called a *flat*. Like the face poset, the poset of flats is also ordered by reverse inclusion, and it is ranked by codimension. Notice that the entire space \mathbb{R}^n is an element of \mathcal{L} (being the intersection of zero hyperplanes), and it is in fact the unique minimal element of \mathcal{L} . Denote by $\mathcal{L}_k(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$ the set of flats of codimension k . The poset of flats is a geometric semilattice, and it is a geometric lattice if \mathcal{A} is central. See Figure 2.2 for an example.

For a subset $U \subseteq \mathbb{R}^n$ (usually a face or a flat), let $\text{supp}(U) \subseteq \mathcal{A}$ be the subarrangement of \mathcal{A} consisting of the hyperplanes that contain U . This is called the *support* of U . Also, denote by $|U| \subseteq \mathbb{R}^n$ the affine span of U . Notice that, for a face $F \in \mathcal{F}$, we have $|F| \in \mathcal{L}$.

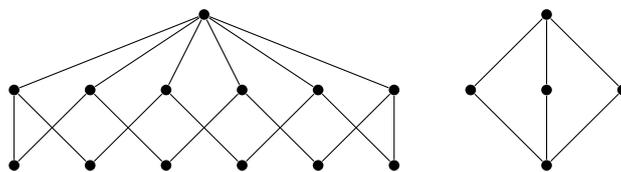


Figure 2.2: Hasse diagrams of the poset of faces (on the left) and of the poset of flats (on the right) of the braid arrangement of type A_2 .

Given a flat $X \in \mathcal{L}$, denote also by \mathcal{A}_X the support of X . Passing from \mathcal{A} to \mathcal{A}_X is an operation called *restriction*. Let $\pi_X: \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}_X)$ be the natural projection, which maps a chamber $C \in \mathcal{C}(\mathcal{A})$ to the unique chamber of \mathcal{A}_X that contains C . Denote by \mathcal{A}^X the arrangement in X given by $\{H \cap X \mid H \notin \mathcal{A}_X\}$. The operation of passing from \mathcal{A} to \mathcal{A}^X is called *contraction*.

2.3 TOPOLOGY OF THE COMPLEMENT

Let \mathcal{A} be a hyperplane arrangement in \mathbb{R}^n . The complement in \mathbb{C}^n of the complexified arrangement $\mathcal{A}_{\mathbb{C}}$ is an important topological space, and we denote it by $M(\mathcal{A})$.

In 1987, Salvetti constructed a regular CW complex with the homotopy type of $M(\mathcal{A})$ [Sal87]. This complex is now known as the *Salvetti complex* of \mathcal{A} . In this section we are going to recall the poset of cells of the Salvetti complex. See also [GR89, BZ92, Sal94, OT13].

For a chamber $C \in \mathcal{C}$ and a face $F \in \mathcal{F}$, denote by $C.F$ the unique chamber $C' \preceq F$ such that $\pi_{|F|}(C) = \pi_{|F|}(C')$. In other words, this is the unique chamber containing F and lying in the same chamber as C in $\mathcal{A}_{|F|}$. The Salvetti complex has a k -cell $\langle C, F \rangle$ for each pair (C, F) , where $C \in \mathcal{C}$ is a chamber and $F \in \mathcal{F}$ is a face of C of codimension k . A cell $\langle C, F \rangle$ is in the boundary of $\langle D, G \rangle$ if and only if $F \prec G$ and $D.F = C$.

Theorem 2.3.1 (Salvetti [Sal87]). The poset $\text{Sal}(\mathcal{A})$ defined above is the poset of cells of a regular CW complex $\mathcal{S}(\mathcal{A})$ with the homotopy type of $M(\mathcal{A})$.

In particular, the Salvetti complex has one 0-dimensional cell for every chamber $C \in \mathcal{C}$, and two 1-dimensional cells for every face $F \in \mathcal{F}$ of codimension 1. In general, the closure of a cell $\langle C, F \rangle$ forms a polytope which is dual to the arrangement $\mathcal{A}_{|F|}$. See Figure 2.3 for an example.

An arrangement \mathcal{A} is called a $K(\pi, 1)$ arrangement if the complement $M(\mathcal{A})$ is a $K(\pi, 1)$ space (i.e. if its universal cover is contractible). Braid arrangements are one of the first non-trivial families of arrangements which were proved to be $K(\pi, 1)$ [FN62a]. Brieskorn conjectured that this property holds for all

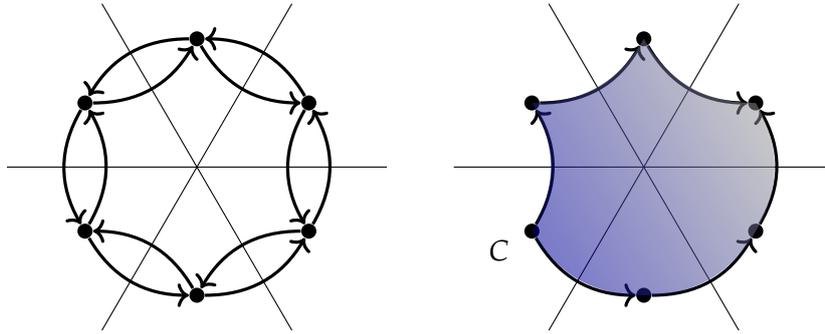


Figure 2.3: On the left, the 1-skeleton of the Salvetti complex of the braid arrangement of type A_2 . On the right, one of the six 2-cells with its boundary.

finite reflection arrangements [Bri73] (see Section 2.4). Shortly after, Deligne proved the following stronger result.

Theorem 2.3.2 (Deligne [Del72]). Let \mathcal{A} be a finite central arrangement. If all the chambers of \mathcal{A} are simplicial cones, then \mathcal{A} is $K(\pi, 1)$.

2.4 COXETER GROUPS

We are now going to introduce Coxeter groups. In our brief exposition, we follow [Bou68, Hum92, Dav08].

Let S be a finite set, and let $M = (m_{s,t})_{s,t \in S}$ be a square matrix indexed by S and satisfying the following properties:

- $m_{s,t} \in \{2, 3, \dots\} \cup \{\infty\}$ for all $s \neq t$, and $m_{s,s} = 1$;
- $m_{s,t} = m_{t,s}$.

Such a matrix is called a *Coxeter matrix*. From a Coxeter matrix M we can construct an edge-labelled graph Γ , called the *Coxeter graph* of M , as follows:

- take S as the set of vertices;
- an edge connects vertices s and t if and only if $m_{s,t} \geq 3$, and this edge is labelled by $m_{s,t}$.

When $m_{s,t} = 3$, the label on the corresponding edge is usually omitted. Sometimes we also say that the pair (Γ, S) is a Coxeter graph, to underline that S is the set of vertices of Γ .

Definition 2.4.1. Let (Γ, S) be a Coxeter graph. The *Coxeter system* of (Γ, S) is the pair (W, S) , where W is the group defined by

$$W = \langle S \mid (st)^{m_{s,t}} = 1 \quad \forall s, t \in S \text{ such that } m_{s,t} \neq \infty \rangle.$$

Figure 2.4: Coxeter graph of type A_n .

A group W as above is called a *Coxeter group*.

An important example is given by the family of Coxeter graphs of type A_n , shown in Figure 2.4. The corresponding Coxeter groups are the symmetric groups \mathfrak{S}_{n+1} .

Definition 2.4.2. Let (W, S) be a Coxeter system. For any $T \subseteq S$, let W_T be the subgroup of W generated by T . A subgroup constructed in this way is called a *standard parabolic subgroup* of W . The pair (W_T, T) is also a Coxeter system, with $\Gamma|_T$ as its associated Coxeter graph [Hum92, Theorem 5.5].

A Coxeter system (W, S) is *irreducible* if the corresponding Coxeter graph Γ is connected. Two generators $s, t \in S$ commute if they belong to different connected components of the Coxeter graph Γ . For this reason, the study of a Coxeter group can be essentially reduced to the study of its maximal irreducible parabolic subgroups.

Coxeter groups admit a faithful representation as groups generated by (not necessarily orthogonal) linear reflections in some real vector space \mathbb{R}^n . By *reflection*, we mean an endomorphism of \mathbb{R}^n of order 2 that pointwise fixes some hyperplane.

Theorem 2.4.3 ([Hum92, Chapter 5]). Let (W, S) be a Coxeter system, with $n = |S|$. There exist a central hyperplane arrangement \mathcal{A} in \mathbb{R}^n , not necessarily locally finite, and a faithful representation $\rho: W \rightarrow \text{GL}(\mathbb{R}^n)$, satisfying the following properties.

- (i) The elements of S (and their conjugates) act as linear reflections with respect to some hyperplane of \mathcal{A} .
- (ii) W acts freely on the complement of \mathcal{A} .
- (iii) Fix a (closed) chamber $C \in \mathcal{C}(\mathcal{A})$. The union $U = \bigcup_{w \in W} w(C)$ is a W -invariant convex cone (called the *Tits cone*).
- (iv) C is a fundamental domain for the action of W on U (the W -orbit of each point of U meets C in exactly one point).

An arrangement \mathcal{A} as in Theorem 2.4.3 is said to be a *reflection arrangement* associated to the Coxeter system (W, S) . If W is finite, then the Tits cone is the whole space \mathbb{R}^n , and the reflections are orthogonal.

Infinite Coxeter groups which can be realized as discrete groups of affine isometries of \mathbb{R}^m , and are generated by affine orthogonal reflections, are called

affine Coxeter groups. The Tits cone of an irreducible affine Coxeter group W is an open half-space with the origin added:

$$U \cong \mathbb{R}^{n-1} \times \mathbb{R}^+ \cup \{0\}.$$

Then the hyperplane $\mathbb{R}^{n-1} \times \{1\}$ is W -invariant, and the restriction of the representation of Theorem 2.4.3 to this hyperplane yields an affine faithful representation such that W acts cocompactly and is generated by affine orthogonal reflections.

Irreducible spherical and affine Coxeter groups are completely classified [Hum92, Chapters 2 and 4]. The corresponding Coxeter graphs are shown in Figures 2.6, 2.7, and 2.8, at the end of this chapter.

2.5 ROOT SYSTEMS

Finite Coxeter groups are closely related to root systems. We introduce root systems following the exposition of Humphreys [Hum92].

Definition 2.5.1 ([Hum92, Sections 1.2 and 2.9]). A finite set Φ of nonzero vectors in \mathbb{R}^n is a *root system* if:

- (i) $\Phi \cap \mathbb{R}\alpha \in \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$;
- (ii) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$, where s_α denotes the orthogonal reflection with respect to the hyperplane orthogonal to α .

A root system Φ is *crystallographic* if it satisfies the following additional requirement:

- (iii) $\frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Every root system has an associated finite Coxeter group, generated by the reflections with respect to the roots. Coxeter groups associated to crystallographic root systems have the additional property of stabilizing a lattice in \mathbb{R}^n . A subgroup of $\text{GL}(\mathbb{R}^n)$ that stabilizes a lattice is called a *crystallographic group*. A Coxeter group (W, S) is crystallographic if and only if $m_{s,t} \in \{2, 3, 4, 6\}$ for all $s \neq t$ in S [Hum92, Section 2.8]. Therefore most of the irreducible Coxeter groups are crystallographic, namely: $A_n, B_n, D_n, E_6, E_7, E_8, F_4, I_2(6)$ (cf. Figure 2.6).

It turns out that for each irreducible crystallographic Coxeter group there is exactly one associated crystallographic root system, with one exception: a Coxeter group of type B_n is associated to two crystallographic root systems, which are denoted by B_n and C_n [Hum92, Section 2.9]. All other irreducible crystallographic root systems are named as the corresponding Coxeter group, except that the root system associated to the group $I_2(6)$ is usually denoted by G_2 .

The four infinite families of irreducible crystallographic root systems can be constructed as follows (see [Hum92, Section 2.10]).

Type A_n . In \mathbb{R}^{n+1} , or in $\mathbb{R}^n = \{x_1 + \cdots + x_n\} \subseteq \mathbb{R}^{n+1}$, take the $n(n+1)$ vectors of the form $e_i - e_j$ for $i \neq j$ (where e_1, \dots, e_{n+1} is the standard basis of \mathbb{R}^{n+1}).

Type B_n . In \mathbb{R}^n , take the $2n$ “short” vectors $\pm e_i$ (for $1 \leq i \leq n$) and the $2n(n-1)$ “long” vectors $\pm e_i \pm e_j$ (for $1 \leq i < j \leq n$).

Type C_n . In \mathbb{R}^n , take the $2n$ “long” vectors $\pm 2e_i$ (for $1 \leq i \leq n$) and the $2n(n-1)$ “short” vectors $\pm e_i \pm e_j$ (for $1 \leq i < j \leq n$).

Type D_n . In \mathbb{R}^n , take the $2n(n-1)$ vectors $\pm e_i \pm e_j$ (for $1 \leq i < j \leq n$).

All irreducible affine Coxeter groups (Figures 2.7 and 2.8) can be constructed from the irreducible crystallographic root systems, as follows (see [Hum92, Chapter 4]). For each α in a crystallographic root system $\Phi \subseteq \mathbb{R}^n$, and for each integer $k \in \mathbb{Z}$, consider the affine hyperplane $H_{\alpha,k} = \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = k\}$. Then take the group generated by the affine orthogonal reflections with respect to the hyperplanes $H_{\alpha,k}$.

2.6 ARTIN GROUPS

Every Coxeter group (W, S) has an associated *Artin group* defined as follows:

$$G_W = \langle g_s, s \in S \mid \underbrace{g_s g_t g_s \cdots}_{m_{s,t} \text{ factors}} = \underbrace{g_t g_s g_t \cdots}_{m_{s,t} \text{ factors}} \rangle.$$

Historically, the first family of Artin groups to be introduced was that of *braid groups* [FN62a, FN62b, Arn69]. The braid group on n strands can be defined as the Artin group of type A_{n-1} . It arises as the fundamental group of the configuration space of n indistinguishable points in the plane:

$$G_{\mathfrak{S}_n} = \pi_1 \left(\left(\mathbb{C}^n \setminus \bigcup_{i \neq j} \{x_i = x_j\} \right) / \mathfrak{S}_n \right).$$

This construction was generalized to every Artin group G_W as follows [Bou68, Vin71, VdL83, GP12, Par14]. Let \mathcal{A} be the reflection arrangement associated to W , and let $U \subseteq \mathbb{R}^n$ be its Tits cone (cf. Theorem 2.4.3). Define a configuration space associated to W as follows:

$$Y = (\text{int}(U) + i\mathbb{R}^n) \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}.$$

Then G_W is the fundamental group of the orbit configuration space $Y_W = Y/W$ [VdL83].

Artin groups corresponding to finite Coxeter groups are called of *spherical type* (or simply *spherical*). Artin groups corresponding to affine Coxeter groups are called of *affine type* (or simply *affine*).

In Section 2.3 we mentioned Deligne’s theorem, which implies that the orbit configuration space Y_W is a $K(G_W, 1)$ if W is finite. This allows to compute homology and cohomology of Artin groups of spherical type by looking at the space Y_W . A famous conjecture by Arnol’d, Pham, and Thom (cf. [VdL83]) states that the same property holds for all Artin groups.

Conjecture 2.6.1 ($K(\pi, 1)$ conjecture for Artin groups). The orbit configuration space Y_W is a $K(G_W, 1)$, for every Coxeter group W .

Much effort was put in proving this conjecture. Apart from the case of spherical Artin groups, which was settled in the work of Deligne [Del72], the $K(\pi, 1)$ conjecture was proved in the following cases.

- Artin groups of dimension ≤ 2 , and of FC type [Hen85, CD95]; these include the so called *right angled Artin groups*, which are defined by Coxeter matrices satisfying $m_{s,t} \in \{2, \infty\}$ for all $s, t \in S$.
- Affine artin groups of type \tilde{A}_n and \tilde{C}_n [Oko79] (see also [CP03]), of type \tilde{B}_n [CMS10], and of type \tilde{G}_2 and \tilde{I}_1 (because they have dimension ≤ 2).

In particular, the conjecture is open for one infinite family of affine Artin groups, namely \tilde{D}_n , and for most of the exceptional affine Artin groups. In a series of recent works [BM15, McC15, MS17], McCammond and others have made significant steps towards a unified treatment of affine Artin groups. In particular, they proved that affine Artin groups admit a finite-dimensional classifying space. In Chapter 9 we relate their construction with the orbit space Y_W , and we show that every affine Artin group admits a finite classifying space. In addition, we prove the $K(\pi, 1)$ conjecture in the case \tilde{D}_4 .

The orbit space Y_W was shown to be homotopy equivalent to the classifying space of another important object associated to G_W , namely its *Artin monoid* [Dob06, Ozo17, Pao17a]. As a consequence, the $K(\pi, 1)$ conjecture carries information about how an Artin group is related to its Artin monoid.

A finite Coxeter group W has an associated finite reflection arrangement \mathcal{A} . Then the configuration space Y has the homotopy type of the Salvetti complex $\mathcal{S}(\mathcal{A})$ (introduced in Section 2.3). The action of W on $\mathcal{S}(\mathcal{A})$ is cellular, and therefore the orbit configuration space Y_W also has the homotopy type of a finite CW complex $X_W = \mathcal{S}(\mathcal{A})/W$. This construction was generalized by Salvetti to all Coxeter groups.

Theorem 2.6.2 ([Sal94, Theorem 1.4]). The orbit configuration space Y_W deformation retracts onto a finite CW complex X_W , given by a union of convex polytopes with explicit identification of their faces.

The cells of the complex X_W are indexed (in a dimension-preserving way) by the faces of the simplicial complex

$$\mathcal{K}_W = \{\sigma \subseteq S \mid \text{the parabolic subgroup } W_\sigma \text{ is finite}\}.$$

We shall refer to X_W as the Salvetti complex of W .

2.7 ABELIAN ARRANGEMENTS

In the last decade there has been a growing interest in other kinds of arrangements, for example arrangements of hypersurfaces in complex tori or in products of elliptic curves. A general definition, which includes the previous ones as well as some of the “classical” hyperplane arrangements, is the following.

Definition 2.7.1 (Abelian arrangement). Consider a (topological) abelian group G , a list of integer vectors $a_1, \dots, a_m \in \mathbb{Z}^n$, and a list of elements $b_1, \dots, b_m \in G$. The *abelian arrangement* associated to these data is the set $\{K_1, \dots, K_m\}$ of subsets of G^n defined by

$$K_i = \left\{ (x_1, \dots, x_n) \in G^n \mid \sum_{j=1}^n a_{ij}x_j = b_i \right\}.$$

If $G = \mathbb{K}$ is a field, we obtain classical hyperplane arrangements (with integer equations). For $G = S^1$ or $G = \mathbb{C}^*$, we obtain the so-called *toric arrangements* (on a compact torus $(S^1)^n$ or on a complex torus $(\mathbb{C}^*)^n$, respectively). If G is an elliptic curve, we obtain the so-called *elliptic arrangements*.

Apart from hyperplane arrangements, the first class of abelian arrangements to be introduced and studied was that of toric arrangements [Loo93, DCP05, DCP10, DCPV10]. As pointed out by De Concini and Procesi [DCP10], there are great similarities between the theory of toric arrangements and the theory of arrangements of affine hyperplanes. Indeed, a (finite) toric arrangement in $(S^1)^n$ or $(\mathbb{C}^*)^n$ can be lifted to an infinite periodic arrangement of affine hyperplanes in \mathbb{R}^n or \mathbb{C}^n . The theory of toric arrangements experienced a great development in both topology and combinatorics. From a topological point of view, important contributions came from the works of Looijenga [Loo93], d’Antonio and Delucchi [dD12, dD15], De Concini and Procesi [DCP05], Callegaro, D’Adderio, Delucchi, Migliorini, and Pagaria [CDD⁺18]. In particular, the work of d’Antonio and Delucchi on minimality of toric arrangements [dD15] is related to the topics of Part II. From a combinatorial point of view, the works of Moci, D’Adderio, and Brändén [Moc12, DM13, BM14] initiated the study of arithmetic Tutte polynomials and arithmetic matroids (a toric analogue of matroids). Delucchi, Riedel, and D’Alì [DR18, DD18] extended this theory by viewing arithmetic matroids as quotients of semimatroids by group actions. Also, a work of Moci on toric

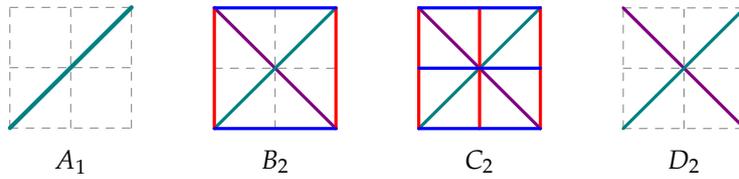


Figure 2.5: Toric arrangements defined by root systems in $S^1 \times S^1$.

arrangements defined by root systems [Moc08] led the way to a more systematic study of the poset of layer of toric arrangements, and to its relations with cohomology [ERS09, Law11, Ber16, Bib18].

In Part IV we are going to contribute to the combinatorial study of the poset of layer of abelian arrangements, focusing on the case of arrangements defined by root systems.

Definition 2.7.2 (Poset of layers). The *poset of layers* $\mathcal{L}(\mathcal{A})$ of an abelian arrangement \mathcal{A} is the set of connected components of all possible intersections of elements of \mathcal{A} . It is ordered by reverse inclusion, and ranked by codimension.

In the case of hyperplane arrangements, intersections of hyperplanes are connected, and therefore the poset of layers coincides with the poset of flats (see Section 2.2). For general abelian arrangements, however, intersections need not be connected.

The poset of flats of a hyperplane arrangement is a geometric lattice, and therefore it is EL-shellable (see Section 1.5). Already for toric arrangements, it is not known in general if the poset of layers is shellable. D’Alì and Delucchi recently showed that this poset is Cohen-Macaulay [DD18], thus providing strong evidence towards the shellability conjecture. In Chapter 7 we are going to show that the posets of layers of toric and elliptic arrangements defined by root systems are EL-shellable. In fact, these posets come from a more general family of posets introduced by Bibby and Gadish [BG18], which we show to be EL-shellable in Chapter 8.

We now explain how root systems give rise to abelian arrangements (see [Bib18, Section 2.2]). Given a crystallographic root system of type A , B , C , or D (see Section 2.5), we can consider the associated abelian arrangement defined by the integer vectors $a_1, \dots, a_m \in \mathbb{Z}^n$ equal to the roots. We take $b_i = 0$ for all i . Notice that the arrangements defined by the root systems of types A_{n-1} , B_n , and D_n are subarrangements of the arrangement defined by the root system of type C_n .

Toric arrangements defined by root systems in $S^1 \times S^1$ are shown in Figure 2.5. Notice that S^1 has two 2-torsion points (namely 1 and -1). Therefore the subset $K \subseteq S^1 \times S^1$ corresponding to the root $2e_i$ is a union of two subtori: $\{x_i = 1\}$ and $\{x_i = -1\}$. Similarly, an elliptic curve has four 2-torsion points.

Then, in an elliptic arrangement, the root $2e_i$ gives rise to a subspace with four connected components.

If \mathcal{A} is a toric or elliptic arrangement, then \mathcal{A} can be seen as the quotient of an infinite arrangement of hyperplanes \mathcal{A}^\dagger in \mathbb{C}^n by a group Γ of translations. We have $\Gamma \cong \mathbb{Z}^n$ for toric arrangements, and $\Gamma \cong \mathbb{Z}^{2n}$ for elliptic arrangements. In any case there is an action of Γ on the semimatroid defined by \mathcal{A}^\dagger , to which in [DR18] is associated a *Tutte polynomial* $T_{\mathcal{A}}(x, y)$. In addition, by [DR18, Theorem F], the characteristic polynomial of $\mathcal{L}(\mathcal{A})$ can be computed as an evaluation of the Tutte polynomial:

$$\chi_{\mathcal{L}(\mathcal{A})}(t) = (-1)^d T_{\mathcal{A}}(1 - t, 0).$$

2. HYPERPLANE ARRANGEMENTS, COXETER GROUPS, ARTIN GROUPS

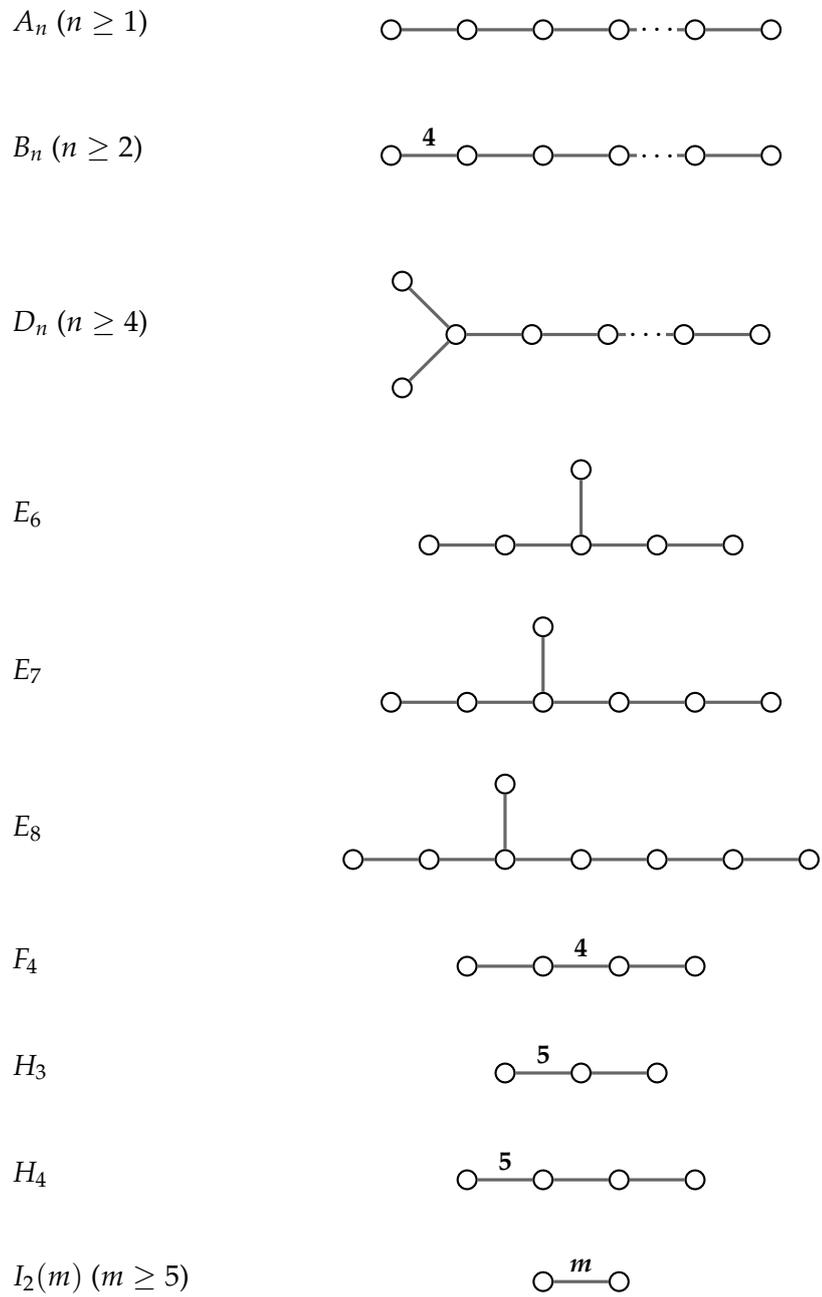


Figure 2.6: Coxeter graphs corresponding to finite Coxeter groups. The subscript equals the number of nodes.

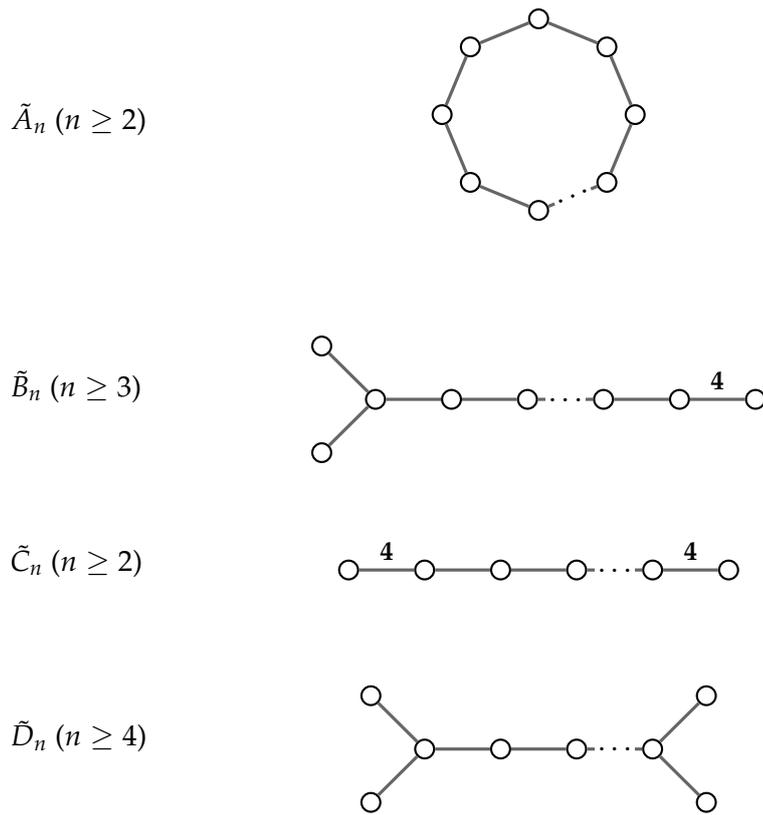


Figure 2.7: Coxeter graphs corresponding to the four infinite families of affine Coxeter groups. The subscript is one less than the number of nodes.

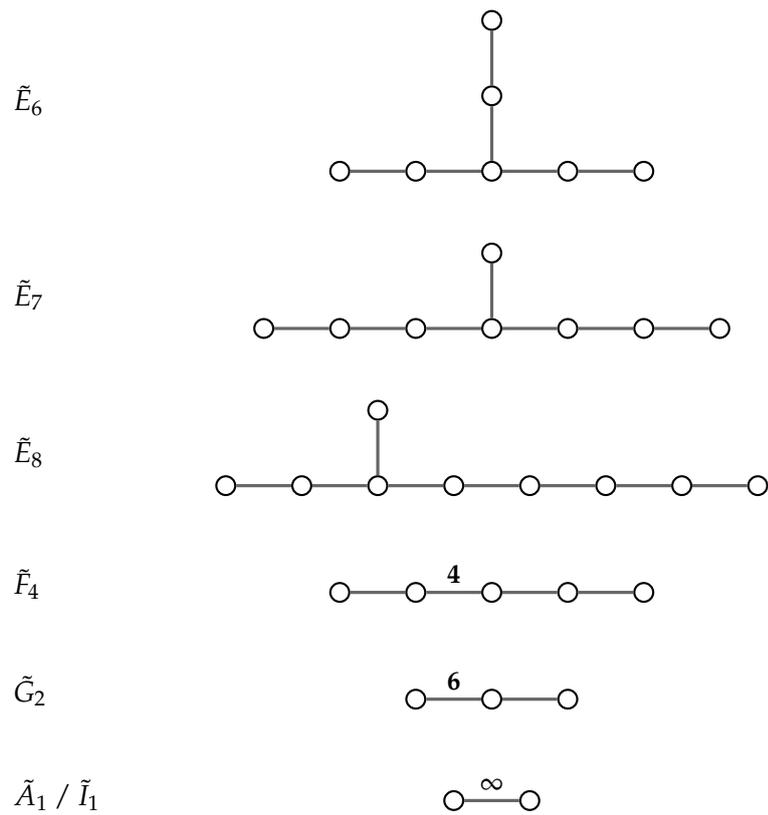


Figure 2.8: Coxeter graphs corresponding to exceptional affine Coxeter groups.

PART II

MINIMALITY

Acyclic matchings on the Salvetti complex

3.1 INTRODUCTION

The material of the next two chapters is based on a joint work with Davide Lofano [LP18].

Let \mathcal{A} be a locally finite arrangement of affine hyperplanes in \mathbb{R}^n . The complement $M(\mathcal{A}) \subseteq \mathbb{C}^n$ of the complexified arrangement $\mathcal{A}_{\mathbb{C}}$ has the homotopy type of the Salvetti complex of \mathcal{A} (see Section 2.3).

For a finite arrangement \mathcal{A} , in [Ran02, DP03, Yos07] it was proved that the complement $M(\mathcal{A})$ has the homotopy type of a minimal CW complex, i.e. with a number of k -cells equal to the k -th Betti number. This minimality result was later made more explicit with discrete Morse theory, in [SS07] (for finite affine arrangements), [Del08] (for finite central arrangements and oriented matroids in general), [GS09] (for finite line arrangements), [dD15] (for affine arrangements with a finite number of directions).

Here we consider a (possibly infinite) affine arrangement \mathcal{A} , and construct a minimal CW model for the complement $M(\mathcal{A})$. This is obtained applying discrete Morse theory to the Salvetti complex of \mathcal{A} . For a (possibly infinite) CW complex, by “minimal” we mean that all the incidence numbers vanish. As in the well known case of finite arrangements, we obtain a geometrically meaningful bijection between cells in the minimal CW model and chambers of \mathcal{A} .

Our starting point is the work of Delucchi on the minimality of oriented matroids [Del08]. Specifically, we build on the idea of decomposing the Salvetti complex according to some “good” total order of the chambers. For a general affine arrangement, however, the combinatorial order used in [Del08] does not yield a decomposition with the desired properties. In Section 3.2

we introduce a class of total orders of the chambers for which we are able to extend the construction of Delucchi, and we call them *valid orders*. We remark that in [Del08, Question 4.18] it was explicitly asked for one such extension to affine arrangements. For a finite affine arrangement, the polar order of Salvetti and Settepanella [SS07] is valid (Remark 3.2.7). Therefore our work contributes to linking the constructions of [SS07] and [Del08] (see also [Del08, Remark 3.8]).

In Section 3.3 we show how to construct an acyclic matching on the Salvetti complex, for any given valid order.

Theorem 3.3.10. Let \mathcal{A} be a locally finite hyperplane arrangement, with a given valid order on the set of chambers. Then there exists a proper acyclic matching on $\text{Sal}(\mathcal{A})$ with critical cells in bijection with the chambers.

We also prove the following two results. The first one can be regarded as a generalization of [Del08, Theorem 3.6], and the second one is the analogue of [Del08, Theorem 3.6], where (finite) oriented matroids are replaced by locally finite hyperplane arrangements.

Theorem 3.3.9. Let X be a k -dimensional polytope in \mathbb{R}^k , and let $y \in \mathbb{R}^k$ be a point outside X that does not lie in the affine hull of any facet of X . Then there exists an acyclic matching on the poset of faces of X visible from y , such that no face is critical.

Theorem 3.4.1. Let \mathcal{A} be a locally finite hyperplane arrangement. For every chamber $C \in \mathcal{C}(\mathcal{A})$, there is a proper acyclic matching on the poset of faces $\mathcal{F}(\mathcal{A})$ such that C is the only critical face.

In Chapter 4 we are going to construct a large class of valid orders, for any locally finite hyperplane arrangement. We are then going to show that the resulting Morse complexes are minimal.

3.2 DECOMPOSITION OF THE SALVETTI COMPLEX

Our aim is to construct an acyclic matching on the Salvetti complex of a locally finite arrangement \mathcal{A} , with critical cells in explicit bijection with the chambers of \mathcal{A} . Following the ideas of Delucchi [Del08], we want to decompose the Salvetti complex into “pieces” (one piece for every chamber) and construct an acyclic matching on each of these pieces with exactly one critical cell. More formally, we are going to decompose the poset of cells $\text{Sal}(\mathcal{A})$ as a disjoint union

$$\text{Sal}(\mathcal{A}) = \bigsqcup_{C \in \mathcal{C}} N(C),$$

where $\mathcal{C} = \mathcal{C}(\mathcal{A})$, so that every subposet $N(C) \subseteq \text{Sal}(\mathcal{A})$ admits an acyclic matching with one critical cell.

Definition 3.2.1. Given a chamber $C \in \mathcal{C}$, let $S(C) \subseteq \text{Sal}(\mathcal{A})$ be the set of all the cells $\langle C', F \rangle \in \text{Sal}(\mathcal{A})$ such that $C' = C.F$. In other words, a cell is in $S(C)$ if all the hyperplanes in $\text{supp}(F)$ do not separate C and C' .

The cells in $S(C)$ form a subcomplex of the Salvetti complex. This subcomplex is dual to the stratification of \mathbb{R}^n induced by \mathcal{A} . Also, the natural map $S(C) \rightarrow \mathcal{F}$ which sends $\langle C', F \rangle$ to F is a poset isomorphism.

Now fix a total order \dashv of the chambers:

$$\mathcal{C} = \{C_0 \dashv C_1 \dashv C_2 \dashv \dots\}$$

(when \mathcal{C} is infinite, the order type is that of natural numbers).

Definition 3.2.2. For every chamber $C \in \mathcal{C}$, let $N(C) \subseteq S(C)$ be the subset consisting of all the cells not included in any $S(C')$ with $C' \dashv C$.

The union of the subcomplexes $S(C)$, for $C \in \mathcal{C}$, is the entire complex $\text{Sal}(\mathcal{A})$. Then the subsets $N(C)$, for $C \in \mathcal{C}$, form a partition of $\text{Sal}(\mathcal{A})$. All the 0-cells are contained in $N(C_0) = S(C_0)$. Therefore, for $C \neq C_0$, the cells of $N(C)$ do not form a subcomplex of the Salvetti complex. If \mathcal{A} is a (finite) central arrangement, this definition of $N(C)$ coincides with the one given in [Del08, Section 4].

We want now to choose the total order \dashv of the chambers so that each $N(C)$ admits an acyclic matching with exactly one critical cell. In [Del08], this is done by taking any linear extension of the partial order \leq_{C_0} , defined as follows: $D' \leq_{C_0} D$ if and only if $s(C_0, D') \subseteq s(C_0, D)$ (here C_0 is any fixed chamber). In the language of oriented matroids, $(\mathcal{C}, \leq_{C_0})$ is called the *tope poset based at C_0* [BLVS⁺99, Definition 4.2.9]. A linear extension of \leq_{C_0} works well for central arrangements but not for general affine arrangements, as we see in the following two examples.

Example 3.2.3. Consider a non-central arrangement of three lines in the plane, as in Figure 3.1 on the left. Choose C_0 as one of the three simplicial unbounded chambers. In any linear extension of \leq_{C_0} , the last chamber C_6 must be the non-simplicial unbounded chamber opposite to C_0 . However, $S(C_6) \subseteq \bigcup_{C \neq C_6} S(C)$, so $N(C_6)$ is empty, and therefore it does not admit an acyclic matching with one critical cell. Figure 3.2 shows the decomposition of the Salvetti complex for one of the possible linear extensions of \leq_{C_0} .

Example 3.2.4. Consider the arrangement of five lines depicted in Figure 3.1 on the right. For every choice of a base chamber C_0 and for every linear extension of \leq_{C_0} , there is some chamber C such that $N(C)$ is empty.

We are now going to state a condition on the total order \dashv on \mathcal{C} that produces a decomposition of the Salvetti complex with the desired properties. First recall the following definition from [Del08].

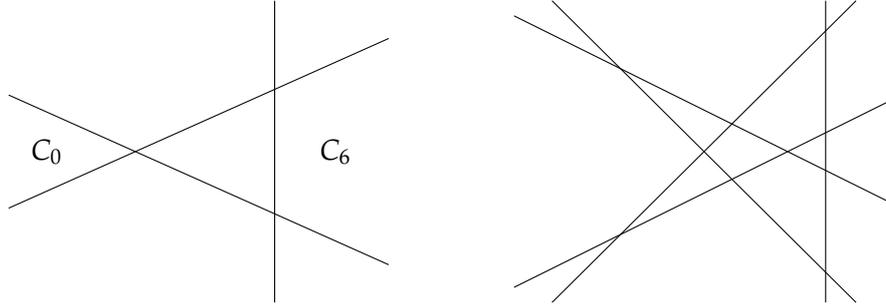


Figure 3.1: Two line arrangements.

Definition 3.2.5. Given a chamber C and a total order \dashv on \mathcal{C} , let

$$\mathcal{J}(C) = \{X \in \mathcal{L} \mid \text{supp}(X) \cap s(C, C') \neq \emptyset \ \forall C' \dashv C\}.$$

Notice that $\mathcal{J}(C)$ is an upper ideal of $\mathcal{L} = \mathcal{L}(\mathcal{A})$, and it coincides with \mathcal{L} for $C = C_0$. In [Del08, Theorem 4.15] it is proved that, if \mathcal{A} is a (finite) central arrangement and \dashv is a linear extension of \leq_{C_0} (for any choice of $C_0 \in \mathcal{C}$), then $\mathcal{J}(C)$ is a principal upper ideal for every chamber $C \in \mathcal{C}$. This is the condition we need.

Definition 3.2.6 (Valid order). A total order \dashv on \mathcal{C} is *valid* if, for every chamber $C \in \mathcal{C}$, $\mathcal{J}(C)$ is a principal upper ideal generated by some flat $X_C = |F_C| \in \mathcal{L}$, where F_C is a face of C .

The total orders of Example 3.2.3 are not valid, because $\mathcal{J}(C_6)$ is empty. A valid order that begins with the chamber C_0 of Example 3.2.3 is shown in Figure 3.3.

The previous definition is the starting point of our answer to [Del08, Question 4.18], where it was asked for an extension to affine arrangements. Sections 3.3 and 4.2 will motivate this definition.

Remark 3.2.7. If \mathcal{A} is a finite affine arrangement, the polar order of the chambers defined by Salvetti and Settepanella [SS07, Definition 4.5] is valid. Indeed, $\mathcal{J}(C)$ is a principal upper ideal generated by $X_C = |F_C|$, where F_C is the smallest face of C with respect to the polar order of the faces. Therefore Definition 3.2.6 highlights the link between the constructions of [SS07] and [Del08] (see also [Del08, Remark 3.8]). The results of Section 3.3, if applied to polar orders, give rise to acyclic matchings that are related to the polar matchings of [SS07].

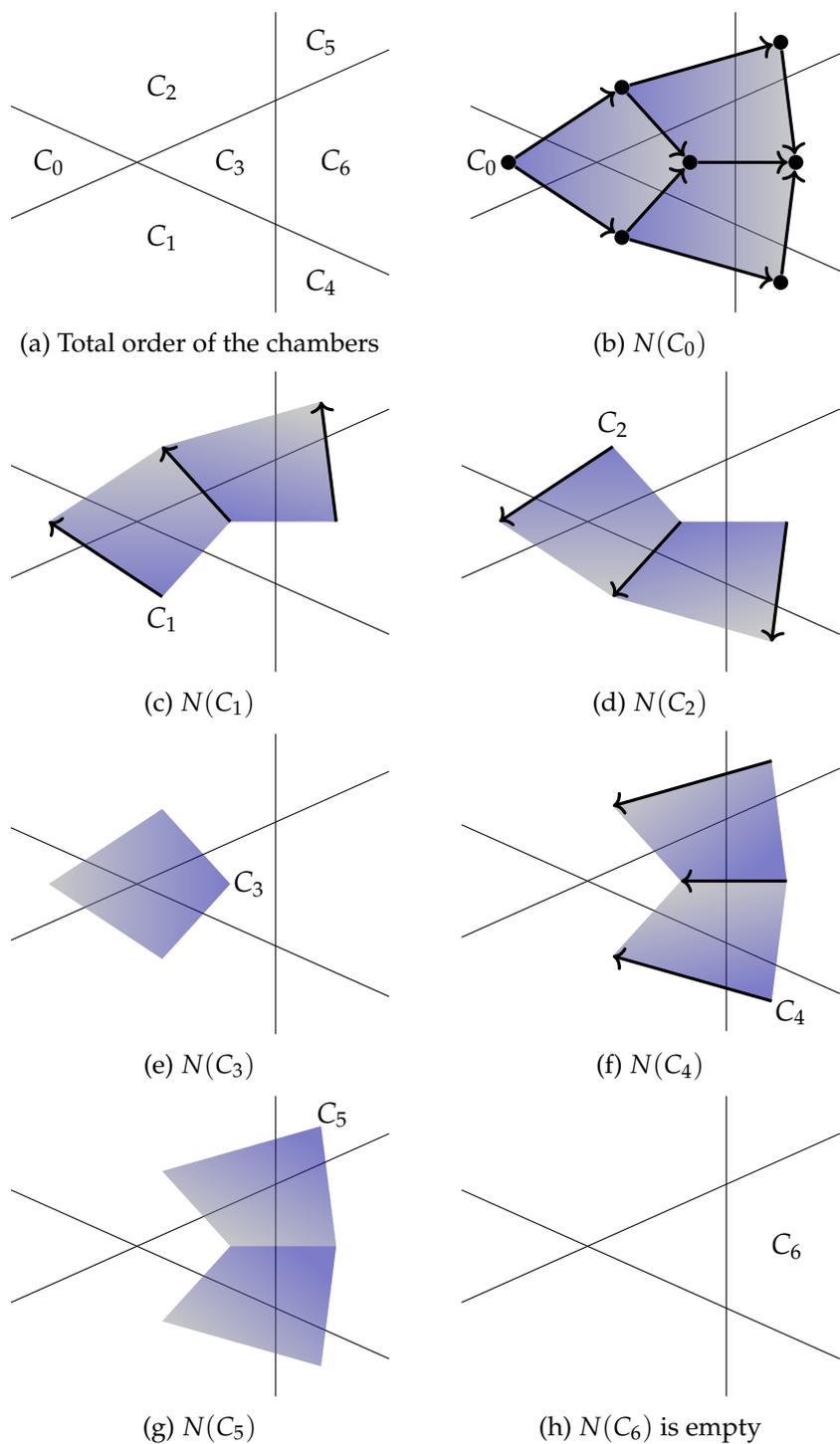


Figure 3.2: A non-central arrangement of three lines in the plane, with a linear extension of \leq_{C_0} . Here $N(C_5)$ and $N(C_6)$ do not admit acyclic matchings with one critical cell.

3. ACYCLIC MATCHINGS ON THE SALVETTI COMPLEX

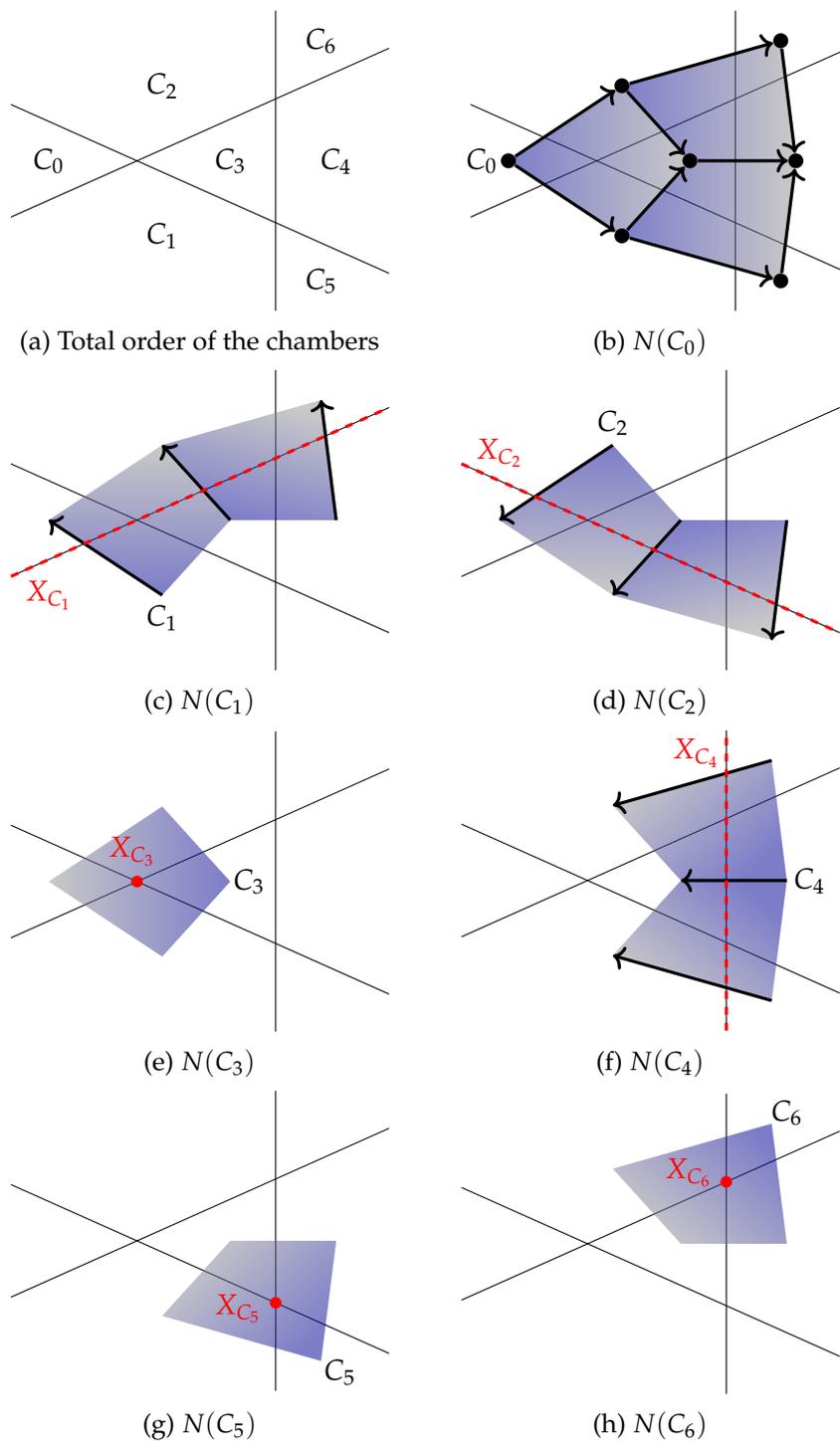


Figure 3.3: A non-central arrangement of three lines in the plane, with a valid order of the chambers. For every chamber C except C_0 , the generator X_C of $\mathcal{J}(C)$ is highlighted.

3.3 ACYCLIC MATCHINGS FROM VALID ORDERS

Throughout this section we assume to have an arrangement \mathcal{A} together with a valid order \dashv of \mathcal{C} (as in Definition 3.2.6). Using the decomposition

$$\text{Sal}(\mathcal{A}) = \bigsqcup_{C \in \mathcal{C}} N(C)$$

of Section 3.2 (induced by the valid order \dashv), we are going to construct a proper acyclic matching on $\text{Sal}(\mathcal{A})$ with critical cells in bijection with the chambers. More precisely, we are going to construct an acyclic matching on every $N(C)$ with exactly one critical cell, and then attach these matchings together using the Patchwork Theorem (Theorem 1.2.6). This strategy is the same as the one employed in [Del08], but our proofs are different since we deal with affine and possibly infinite arrangements.

Lemma 3.3.1. Suppose that \dashv is a valid order on \mathcal{C} . Then

$$N(C) = \{\langle D, F \rangle \in S(C) \mid F \subseteq X_C\}.$$

Proof. To prove the inclusion \subseteq , assume by contradiction that there exists some cell $\langle D, F \rangle \in N(C)$ with $F \not\subseteq X_C$. By minimality of X_C in $\mathcal{J}(C)$, we have that $|F| \notin \mathcal{J}(C)$. This means that there exists a chamber $C' \dashv C$ such that $\text{supp}(F) \cap s(C, C') = \emptyset$. Then C and C' are contained in the same chamber of $\mathcal{A}_{|F|}$, which implies $C'.F = C.F$. By definition of $S(C)$, we have that $C.F = D$. Then $C'.F = D$, so $\langle D, F \rangle \in S(C')$. This is a contradiction, since $\langle D, F \rangle \in N(C)$ and $C' \dashv C$.

For the opposite inclusion, consider a cell $\langle D, F \rangle \in S(C)$ with $F \subseteq X_C$. Then $|F| \in \mathcal{J}(C)$, so for every chamber $C' \dashv C$ there exists an hyperplane in $\text{supp}(F) \cap s(C, C')$. By the same argument as before, we can deduce that $D = C.F \neq C'.F$ for all $C' \dashv C$, which means that $\langle D, F \rangle \notin S(C')$ for all $C' \dashv C$. Therefore $\langle D, F \rangle \in N(C)$. \square

For a chamber $D \in \mathcal{C}$ and a face $F \succeq D$, denote by D^F the chamber opposite to D with respect to F . For every chamber $C \in \mathcal{C}$, consider the map

$$\tilde{\eta}_C: S(C) \rightarrow \mathcal{C}$$

that sends a cell $\langle D, F \rangle$ to D^F .

Lemma 3.3.2. The map $\tilde{\eta}_C: S(C) \rightarrow (\mathcal{C}, \leq_C)$ is order-preserving.

Proof. Let $\langle D, F \rangle, \langle D', F' \rangle \in S(C)$, and suppose that $\langle D', F' \rangle \leq \langle D, F \rangle$ (see Figure 3.4). Then $F' \preceq F$ and therefore $\text{supp}(F') \subseteq \text{supp}(F)$. Call $E = D^F$ and $E' = D'^{F'}$. By definition of $S(C)$, we have that $s(C, E) = s(C, D) \cup \text{supp}(F)$ and $s(C, E') = s(C, D') \cup \text{supp}(F')$. In addition, $F' \preceq F$ implies

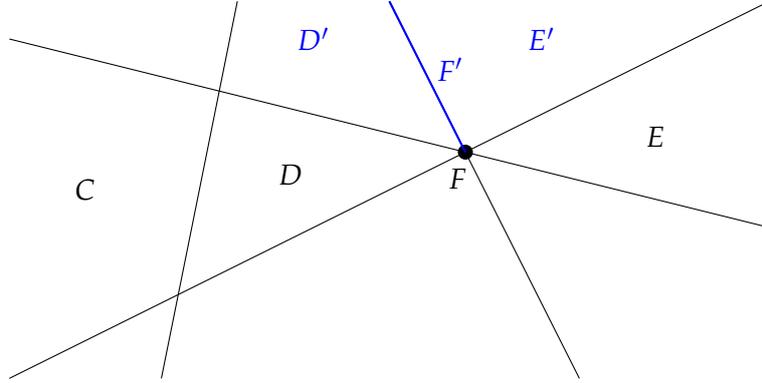


Figure 3.4: Proof of Lemma 3.3.2.

that $s(D, D') \subseteq \text{supp}(F) \setminus \text{supp}(F')$. Since $s(C, D') \subseteq s(C, D) \cup s(D, D')$, we conclude that

$$\begin{aligned} s(C, E') &= s(C, D') \cup \text{supp}(F') \subseteq s(C, D) \cup s(D, D') \cup \text{supp}(F') \\ &\subseteq s(C, D) \cup \text{supp}(F) = s(C, E). \end{aligned}$$

Therefore $E' \leq_C E$. \square

Consider the restriction $\eta_C = \tilde{\eta}_C|_{N(C)} : N(C) \rightarrow \mathcal{C}$. The matching on $N(C)$ will be obtained as a union of acyclic matchings on each fiber $\eta_C^{-1}(E)$ of η_C . Lemma 3.3.2, together with the Patchwork Theorem, will ensure that the matching on $N(C)$ is acyclic. We now fix two chambers C and E , and study the fiber $\eta_C^{-1}(E)$.

Lemma 3.3.3. Let \dashv be a valid order on \mathcal{C} , and let C, E be two chambers. A cell $\langle D, F \rangle \in \text{Sal}(\mathcal{A})$ is in the fiber $\eta_C^{-1}(E)$ if and only if $D = E^F$, $F \subseteq X_C$, and $\text{supp}(F) \subseteq s(C, E)$.

Proof. Suppose that $\langle D, F \rangle \in \eta_C^{-1}(E)$. In particular, $\langle D, F \rangle \in N(C)$, thus by Lemma 3.3.1 we have that $F \subseteq X_C$. By definition of η_C , $D^F = E$ and so $E^F = D$. Finally, we have $\text{supp}(F) \subseteq s(D, E)$ by definition of η_C , and $\text{supp}(F) \cap s(C, D) = \emptyset$ by definition of $S(C)$, so $\text{supp}(F) \subseteq s(D, E) \setminus s(C, D) \subseteq s(C, E)$.

We want now to prove that a cell $\langle D, F \rangle$ that satisfies the given conditions is in the fiber $\eta_C^{-1}(E)$. Since D is opposite to E with respect to F , we deduce that $\text{supp}(F) \subseteq s(D, E)$. Then, using the hypothesis $\text{supp}(F) \subseteq s(C, E)$, we obtain $\text{supp}(F) \cap s(C, D) = \emptyset$. This means that $C.F = D$, i.e. $\langle D, F \rangle \in S(C)$. By Lemma 3.3.1, we conclude that $\langle D, F \rangle \in N(C)$. The fact that $\eta_C(\langle D, F \rangle) = E$ follows directly from the definition of η_C . \square

A cell $\langle D, F \rangle$ in the fiber $\eta_C^{-1}(E)$ is determined by F , because $D = E^F$. Then we immediately have the following corollary.

Corollary 3.3.4. The fiber $\eta_C^{-1}(E)$ is in order-preserving (and rank-preserving) bijection with the set of faces $F \succeq E$ such that $F \subseteq X_C$ and $\text{supp}(F) \subseteq s(C, E)$. In particular, if $\eta_C^{-1}(E)$ is non-empty, then $\text{supp}(X_C) \subseteq s(C, E)$. \square

Assume from now on that the fiber $\eta_C^{-1}(E)$ is non-empty. The above corollary can be restated as follows, restricting to the flat X_C .

Corollary 3.3.5. Suppose that the fiber $\eta_C^{-1}(E)$ is non-empty. Then $C' = C \cap X_C$ and $E' = E \cap X_C$ are chambers of the arrangement \mathcal{A}^{X_C} , and $\eta_C^{-1}(E)$ is in order-preserving bijection with the set of faces $F \succeq E'$ such that $\text{supp}(F) \subseteq s(C', E')$ in \mathcal{A}^{X_C} .

Proof. By Definition 3.2.6, $X_C = |F_C|$ for some face F_C of C . Then $C' = C \cap X_C = F_C$ is a chamber of \mathcal{A}^{X_C} .

Consider now any cell $\langle D, F \rangle \in \eta_C^{-1}(E)$, and let $D' = D \cap X_C$. If we prove that D' is a chamber of \mathcal{A}^{X_C} , then the same is true for E' , since they are opposite with respect to F and $F \subseteq X_C$ (by Lemma 3.3.1). Let $F'_C = F_C.F$ in the arrangement \mathcal{A}^{X_C} (so F'_C is a chamber of \mathcal{A}^{X_C}), and consider the chamber $\tilde{D} = C.F'_C$ in \mathcal{A} . Then $\tilde{D} = C.F = D$, where the first equality holds because $F'_C \preceq F$, and the second equality because $D \in S(C)$. Therefore $D' = D \cap X_C = \tilde{D} \cap X_C = F'_C$ is a chamber of \mathcal{A}^{X_C} .

The second part is mostly a rewriting of Corollary 3.3.4, but some care should be taken since we are passing from the arrangement \mathcal{A} to the arrangement \mathcal{A}^{X_C} . To avoid confusion, in \mathcal{A}^{X_C} write supp' and s' in place of supp and s . Given a face $F \subseteq X_C$, we need to prove that $\text{supp}(F) \subseteq s(C, E)$ in \mathcal{A} if and only if $\text{supp}'(F) \subseteq s'(C', E')$ in \mathcal{A}^{X_C} . This is true because

$$\begin{aligned} \text{supp}'(F) &= \{H \cap X_C \mid H \in \text{supp}(F) \text{ and } H \not\supseteq X_C\} \\ s'(C', E') &= \{H \cap X_C \mid H \in s(C, E) \text{ and } H \not\supseteq X_C\}. \end{aligned} \quad \square$$

Constructing an acyclic matching on $\eta_C^{-1}(E)$ is then the same as constructing an acyclic matching on the set of faces of E' given by Corollary 3.3.5. We start by considering the special case $E' = C'$.

Lemma 3.3.6. Suppose that the fiber $\eta_C^{-1}(E)$ is non-empty. Then $E' = C'$ if and only if E is the chamber opposite to C with respect to X_C . In this case, $\eta_C^{-1}(E)$ contains the single cell $\langle C, F_C \rangle$.

Proof. If E is opposite to C with respect to X_C , then clearly $E' = C'$. Conversely, suppose that $E' = C' = F_C$. Let $\langle D, F \rangle$ be any cell in $\eta_C^{-1}(E)$. As in the proof of Corollary 3.3.5, we have that $D \cap X_C = F'_C$, where $F'_C = F_C.F$ in \mathcal{A}^{X_C} . Notice that $F \subseteq E \cap X_C = E' = F_C$, so $F'_C = F_C.F = F_C$. In other words, the chambers C, D and E all contain the face F_C . Since $F \subseteq F_C \subseteq C \cap D$, we have that $s(C, D) \subseteq \text{supp}(F)$. But $D \in S(C)$ implies that $D = C.F$, i.e. $s(C, D) \cap \text{supp}(F) = \emptyset$. Therefore $s(C, D) = \emptyset$, so $C = D$. Now, E is the opposite of D with respect to F , and $E \cap X_C = D \cap X_C = F_C$, so $F = F_C$. This

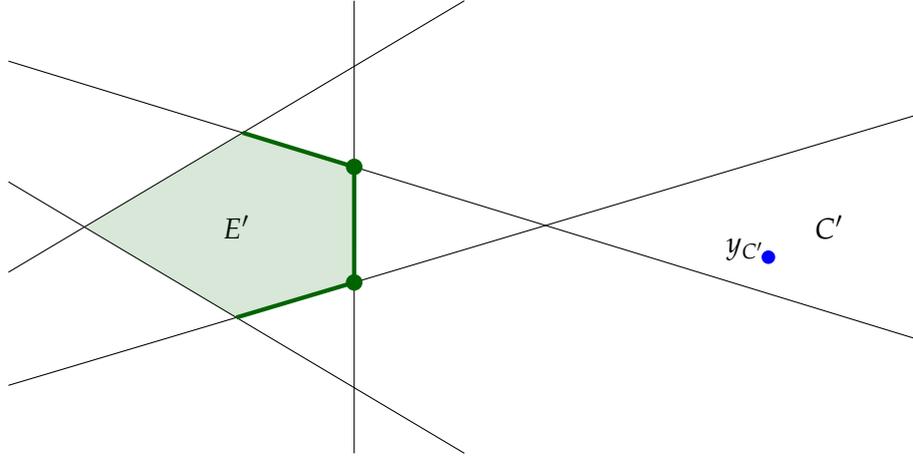


Figure 3.5: The faces of E' that are visible from a point $y_{C'}$ in the interior of C' .

means that E is the opposite of C with respect to X_C . The previous argument also shows that $\eta_C^{-1}(E)$ contains the single cell $\langle C, F_C \rangle$. \square

In particular, for every chamber C there is exactly one fiber $\eta_C^{-1}(E)$ for which $E' = C'$. This fiber contains exactly one cell, which is going to be critical with respect to our matching.

Consider now the case $E' \neq C'$. In view of Corollary 3.3.5, we work with the restricted arrangement \mathcal{A}^{X_C} in X_C . Until Lemma 3.3.8, our notation (for example, $\text{supp}(F)$ and $s(C', E')$) is intended with respect to the arrangement \mathcal{A}^{X_C} . In what follows, we make use of the definitions and facts of Section 1.4.

Lemma 3.3.7. Let $y_{C'}$ be a point in the interior of C' . The faces $F \succeq E'$ such that $\text{supp}(F) \subseteq s(C', E')$ are exactly the faces of E' that are visible from $y_{C'}$.

Proof. Suppose that $\text{supp}(F) \subseteq s(C', E')$. In particular, for every facet $G \supseteq F$ of E' , the hyperplane $|G| \in \mathcal{A}^{X_C}$ separates C' and E' and so G is visible from $y_{C'}$. Then F is visible from $y_{C'}$.

Conversely, suppose that F is visible from $y_{C'}$. Denote by $\mathcal{B} \subseteq \text{supp}(F)$ the set of hyperplanes $|G|$ where $G \supseteq F$ is a facet of E' . All the facets $G \supseteq F$ of E' are visible from $y_{C'}$, so the hyperplanes $|G|$ separate C' and E' . In other words, $\mathcal{B} \subseteq s(C', E')$. In the central arrangement $\mathcal{A}_{|F|}^{X_C} = \text{supp}(F)$, the chambers $\pi_{|F|}(C')$ and $\pi_{|F|}(E')$ are therefore opposite to each other, and \mathcal{B} is the set of their walls. Then every hyperplane in $\text{supp}(F)$ separates C' and E' . \square

Fix an arbitrary point $y_{C'}$ in the interior of C' . By the previous lemma, the faces F given by Corollary 3.3.5 are exactly the faces of E' that are visible from $y_{C'}$. See Figure 3.5 for an example.

The idea now is that, if E' is bounded, then the boundary of E' is shellable, and we can use a shelling to construct an acyclic matching on the set of visible faces. We first need to reduce to the case of a bounded chamber (i.e. a polytope).

Lemma 3.3.8. There exists a finite set \mathcal{A}' of hyperplanes in X_C , and a bounded chamber $\tilde{E} \subseteq E'$ of the hyperplane arrangement $\mathcal{A}' \cup \mathcal{A}^{X_C}$, such that the poset of faces of \tilde{E} that are visible from $y_{C'}$ is isomorphic to the poset of faces of E' that are visible from $y_{C'}$.

Proof. Let $X_C \cong \mathbb{R}^k$. Let Q be a finite set of points which contains $y_{C'}$ and a point in the relative interior of each visible face of E' . For $i = 1, \dots, k$, define $q_i \in \mathbb{R}$ as the minimum of all the i -th coordinates of the points in Q , and q^i as the maximum.

Choose \mathcal{A}' as the set of the $2k$ hyperplanes of the form $\{x_i = q_i - 1\}$ and $\{x_i = q^i + 1\}$, for $i = 1, \dots, k$. Let \tilde{E} be the chamber of $\mathcal{A}^{X_C} \cup \mathcal{A}'$ that contains $Q \setminus \{y_{C'}\}$. By construction, \tilde{E} is bounded and is contained in E' . See Figure 3.6 for an example.

The walls of E' and of \tilde{E} are related as follows: $\mathcal{W}_{\tilde{E}} = \mathcal{W}_{E'} \cup \mathcal{A}''$ for some $\mathcal{A}'' \subseteq \mathcal{A}'$. The hyperplanes in $\mathcal{W}_{E'}$ separate $y_{C'}$ and \tilde{E} , whereas the hyperplanes in \mathcal{A}'' do not. This means that a facet \tilde{G} of \tilde{E} is visible if and only if $|\tilde{G}| \in \mathcal{W}_{E'}$.

There is a natural order-preserving (and rank-preserving) injection φ from the set \mathcal{V} of the visible faces F of E' to the set of faces of \tilde{E} , which maps a face F to the unique face \tilde{F} of \tilde{E} such that $F \cap Q \subseteq \tilde{F} \subseteq F$. We want to show that the image of φ coincides with the set of visible faces of \tilde{E} .

Consider a facet \tilde{G} of \tilde{E} . Then \tilde{G} is in the image of φ if and only if $|\tilde{G}| \notin \mathcal{A}''$, which happens if and only if \tilde{G} is visible.

Consider now a generic face \tilde{F} of \tilde{E} . If $\tilde{F} = \varphi(F)$ for some $F \in \mathcal{V}$, then $Q \cap F \subseteq \tilde{F}$ and so \tilde{F} is not contained in any hyperplane of \mathcal{A}'' . Then all the facets $\tilde{G} \supseteq \tilde{F}$ of \tilde{E} are visible, and so \tilde{F} is visible. Conversely, if \tilde{F} is not in the image of φ , then \tilde{F} is contained in some hyperplane of \mathcal{A}'' and therefore also in some non-visible facet \tilde{G} . Then \tilde{F} is not visible. \square

We now show that the poset of visible faces of a polytope admits an acyclic matching such that no face is critical. We will use this result on the polytope \tilde{E} , in order to obtain a matching on the fiber $\eta_C^{-1}(E)$.

Theorem 3.3.9. Let X be a k -dimensional polytope in \mathbb{R}^k , and let $y \in \mathbb{R}^k$ be a point outside X that does not lie in the affine hull of any facet of X . Then there exists an acyclic matching on the poset of faces of X visible from y such that no face is critical.

Proof. By Theorem 1.4.3 and Lemma 1.4.4, there is a shelling G_1, \dots, G_s of ∂X such that the facets visible from y are the last ones, say G_t, G_{t+1}, \dots, G_s . Notice

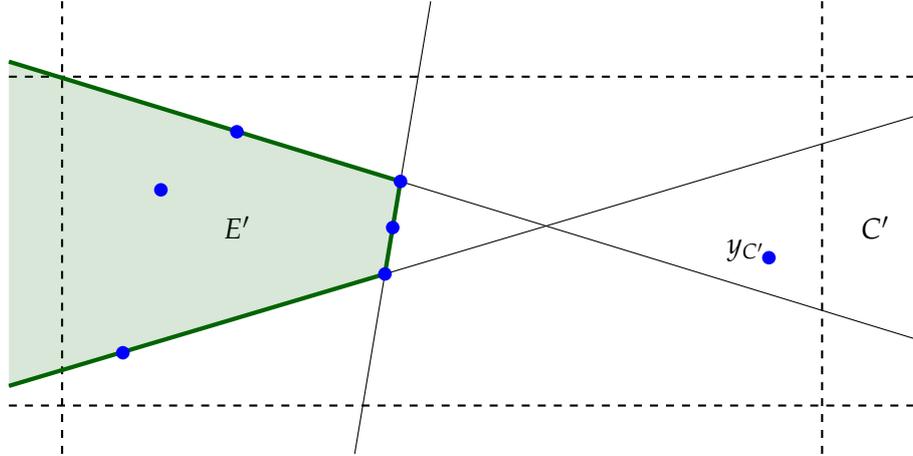


Figure 3.6: Construction of the bounded chamber $\tilde{E} \subseteq E'$ in the proof of Lemma 3.3.8. The points of Q are highlighted, and the hyperplanes of \mathcal{A}' are dashed.

that there is at least one visible facet and at least one non-visible facet. In particular, the first facet G_1 is not visible and the last facet G_s is visible. In other words, we have $2 \leq t \leq s$.

In [Del08, Proposition 1] it is proved that a shelling of a regular CW complex Y induces an acyclic matching on the poset of cells $(P, <)$ of Y (augmented with the empty face \emptyset), with critical cells corresponding to the spanning facets of the shelling. In our case, $Y = \partial X$ is a regular CW decomposition of a sphere, so the only spanning facet of a shelling is the last one (see for example [Del08, Lemma 2.13]).

Let \mathcal{M} be an acyclic matching on ∂X induced by the shelling G_1, \dots, G_s , as in [Del08, Proposition 1]. We claim that the construction of [Del08] produces a matching which is homogeneous with respect to the grading $\varphi: (P, <) \rightarrow \{1, \dots, s\}$ given by

$$\varphi(F) = \min\{i \in \{1, \dots, s\} \mid F \leq G_i\}.$$

To prove this, we need to briefly go through the construction of \mathcal{M} . The first step [Del08, Lemma 2.10] is to construct a total order \sqsubset_i on each P_i (the set of faces of codimension i). The order \sqsubset_0 is simply the shelling order of the facets. It follows from the recursive construction of \sqsubset_i that each $\varphi|_{P_i}: (P_i, \sqsubset_i) \rightarrow \{1, \dots, s\}$ is order-preserving. Then the linear extension \triangleleft of P constructed in [Del08, Definition 2.11] is such that $\varphi: (P, \triangleleft) \rightarrow \{1, \dots, s\}$ is also order-preserving. By construction of the matching [Del08, Lemma 2.12], if $(p, q) \in \mathcal{M}$ (with $p \geq q$) then $p \triangleleft q$. From this we obtain $\varphi(p) \geq \varphi(q)$ and $\varphi(p) \leq \varphi(q)$, so $\varphi(p) = \varphi(q)$. Therefore the matching is homogeneous with respect to φ .

The set of visible faces of X is $\varphi^{-1}(\{t, \dots, s\}) \cup \{X\}$. Notice that the empty face \emptyset belongs to $\varphi^{-1}(1)$, so it does not appear in $\varphi^{-1}(\{t, \dots, s\})$ because $t \geq 2$.

Let \mathcal{M}' be the restriction of \mathcal{M} to $\varphi^{-1}(\{t, \dots, s\})$. This is an acyclic matching on $\varphi^{-1}(\{t, \dots, s\})$ with exactly one critical face, the facet G_s . Then $\mathcal{M}' \cup \{(X, G_s)\}$ is an acyclic matching on the poset of visible faces of X such that no face is critical. \square

We are finally able to attach the matchings on the fibers $\eta_C^{-1}(E)$, using the previous results of this section.

Theorem 3.3.10 (Acyclic matching on $\text{Sal}(\mathcal{A})$). Let \mathcal{A} be a locally finite hyperplane arrangement, and let \preceq be a valid order on the set of chambers $\mathcal{C}(\mathcal{A})$. For every chamber $C \in \mathcal{C}(\mathcal{A})$, there exists a proper acyclic matching on $N(C)$ such that the only critical cell is $\langle C, F_C \rangle$. The union of these matchings forms a proper acyclic matching on $\text{Sal}(\mathcal{A})$ with critical cells in bijection with the chambers.

Proof. Consider the map $\eta: \text{Sal}(\mathcal{A}) \rightarrow \mathcal{C} \times \mathcal{C}$ defined as

$$\eta(\langle D, F \rangle) = (C, D^F),$$

where $C \in \mathcal{C}$ is the chamber such that $\langle D, F \rangle \in N(C)$.

Corollary 3.3.5 provides a description of the non-empty fibers $\eta^{-1}(C, E)$, since by definition we have $\eta^{-1}(C, E) = \eta_C^{-1}(E)$. By Lemma 3.3.6, we know that for every $C \in \mathcal{C}$ there is exactly one non-empty fiber such that $E \cap X_C = C \cap X_C$, and this fiber contains the single cell $\langle C, F_C \rangle$. By Lemma 3.3.7 and Lemma 3.3.8, every other non-empty fiber $\eta^{-1}(C, E)$ is isomorphic to the poset of visible faces of some polytope in X_C (with respect to some external point not lying on the affine hull of the facets). Finally, by Theorem 3.3.9, this poset admits an acyclic matching with no critical faces.

We want to use the Patchwork Theorem (Theorem 1.2.6) to attach these matchings together. To do so, we first need to define a partial order on $\mathcal{C} \times \mathcal{C}$ that makes η a poset map. The order \leq on $\mathcal{C} \times \mathcal{C}$ is the transitive closure of:

$$(C', E') \leq (C, E) \text{ if and only if } C' \preceq C \text{ and } E' \leq_C E$$

(we denote by \preceq the “less than or equal to” with respect to the total order \preceq).

To prove that η is a poset map, suppose to have $\langle D', F' \rangle \leq \langle D, F \rangle$ in $\text{Sal}(\mathcal{A})$. Let $\eta(\langle D', F' \rangle) = (C', E')$ and $\eta(\langle D, F \rangle) = (C, E)$. Since $S(C)$ is a lower ideal of $\text{Sal}(\mathcal{A})$, we immediately obtain that $\langle D', F' \rangle \in S(C)$ and thus $C' \preceq C$. Then, Lemma 3.3.2 implies that $E' \leq_C E$. Therefore $(C', E') \leq (C, E)$.

By the Patchwork Theorem, the union of the matchings on the fibers of η forms an acyclic matching on $\text{Sal}(\mathcal{A})$, with critical cells in bijection with the chambers.

We now need to prove that this matching is proper. To do so, we prove that the $(\mathcal{C} \times \mathcal{C})$ -grading η is compact. Since every fiber $\eta^{-1}(C, E)$ is finite by Lemma 3.3.3, we only need to show that the poset $(\mathcal{C} \times \mathcal{C})_{\leq(C,E)}$ is finite for every pair of chambers (C, E) .

We prove this by double induction, first on the chamber C (with respect to the order \dashv) and then on $m = |s(C, E)|$. The base case, $C = C_0$ and $m = 0$, is trivial since $E = C_0$.

We want now to prove the induction step. Given a pair $(C, m) \in \mathcal{C} \times \mathbb{N}$, suppose that the claim is true for every pair (C', m') such that either $C' \dashv C$, or $C' = C$ and $m' < m$. For every chamber E with $|s(C, E)| = m$ we have that

$$(\mathcal{C} \times \mathcal{C})_{\leq(C,E)} = \bigcup_{\substack{C' \dashv C \\ E' \leq_C E \\ (C', E') \neq (C, E)}} (\mathcal{C} \times \mathcal{C})_{\leq(C', E')} \cup \{(C, E)\}.$$

This is a union of a finite number of sets, and by induction hypothesis every set $(\mathcal{C} \times \mathcal{C})_{\leq(C', E')}$ is finite. Therefore the set $(\mathcal{C} \times \mathcal{C})_{\leq(C,E)}$ is finite.

By the Patchwork Theorem, the matchings on the fibers $\eta^{-1}(C, E)$ can be attached together to form a proper acyclic matching on $\text{Sal}(\mathcal{A})$. By construction, this matching is a union of proper acyclic matchings on the subposets $N(C)$ for $C \in \mathcal{C}$, each of them having $\langle C, F_C \rangle$ as the only critical cell. \square

3.4 REMARKS ON THE POSET OF FACES

We end this chapter with a few remarks. We are not going to use them in Chapter 4, but they are interesting by themselves (especially in relation with [Del08]).

The first remark is that, without the need of a valid order, the results of Section 3.3 allow to obtain a proper acyclic matching on $S(C_0)$ (for any chamber $C_0 \in \mathcal{C}$) with the single critical cell $\langle C_0, C_0 \rangle$. This is because $N(C_0) = S(C_0)$, and in the construction of the matching on $N(C_0)$ we do not use the existence of a valid order that begins with C_0 . As noted in Section 3.2, there is a natural poset isomorphism $S(C_0) \cong \mathcal{F}$ for every chamber $C_0 \in \mathcal{C}$. Then the existence of an acyclic matching on $S(C_0)$ can be stated purely in terms of \mathcal{F} , without speaking of the Salvetti complex. This result appeared in [Del08, Theorem 3.6] in the case of the face poset of an oriented matroid.

Theorem 3.4.1. Let \mathcal{A} be a locally finite hyperplane arrangement. For every chamber $C \in \mathcal{C}(\mathcal{A})$, there is a proper acyclic matching on the poset of faces $\mathcal{F}(\mathcal{A})$ such that C is the only critical face. \square

The second remark is that, given a valid order \dashv of \mathcal{C} and a chamber $C \in \mathcal{C}$, the poset $N(C)$ is isomorphic to $\mathcal{F}(\mathcal{A}^{X_C})$. This is the analogue of [Del08, Lemma 4.20].

Lemma 3.4.2. Suppose that \dashv is a valid order on \mathcal{C} . For every chamber $C \in \mathcal{C}$ there is a poset isomorphism

$$N(C) \cong \mathcal{F}(\mathcal{A}^{X_C}).$$

Proof. The isomorphism in the left-to-right direction sends a cell $\langle D, F \rangle \in N(C)$ to the face F , which is in $\mathcal{F}(\mathcal{A}^{X_C})$ by Lemma 3.3.1. The inverse map sends a face $F \in \mathcal{F}(\mathcal{A}^{X_C})$ to the cell $\langle C.F, F \rangle$, which is in $N(C)$ by definition of $S(C)$ and by Lemma 3.3.1. These maps are order-preserving. \square

Together, Lemma 3.4.2 and Theorem 3.4.1 give an alternative (but equivalent) construction of our matching on $\text{Sal}(\mathcal{A})$, closer to the approach of [Del08].

Euclidean matchings and minimality

4.1 INTRODUCTION

Let \mathcal{A} be a locally finite arrangement of hyperplanes in \mathbb{R}^n . For a given valid order of the chambers, in Chapter 3 we showed how to construct an acyclic matching on the Salvetti complex.

In this chapter we continue to follow [LP18]. In Section 4.2 we construct valid orders for any locally finite arrangement \mathcal{A} , considering the Euclidean distance of the chambers from a fixed generic point $x_0 \in \mathbb{R}^n$. In this way, we obtain a family of matchings on $\text{Sal}(\mathcal{A})$ that we call *Euclidean matchings*. The idea of constructing a minimal complex that depends on a “generic point” appears to be new, as opposed to the more classical approach of using a “generic flag” [Yos07, SS07, GS09]. The critical cells are in bijection with the chambers, and can be described explicitly.

Theorem 4.2.8. Let \mathcal{A} be a locally finite arrangement in \mathbb{R}^n . For every generic point $x_0 \in \mathbb{R}^n$, there exists a Euclidean matching on $\text{Sal}(\mathcal{A})$ with base point x_0 . Such a matching has exactly one critical cell $\langle C, F_C \rangle$ for every chamber $C \in \mathcal{C}(\mathcal{A})$, where F_C is the smallest face of C that contains the projection of x_0 onto C .

In Section 4.3 we prove that, also for infinite arrangements, the Morse complex of a Euclidean matching is minimal.

Theorem 4.3.3. Let \mathcal{A} be a locally finite hyperplane arrangement in \mathbb{R}^n , and let \mathcal{M} be a Euclidean matching on $\text{Sal}(\mathcal{A})$ with base point x_0 . Then the associated Morse complex $\text{Sal}(\mathcal{A})_{\mathcal{M}}$ is minimal (i.e. all the incidence numbers vanish).

In particular, we obtain a new geometric way to read the Betti numbers and the Poincaré polynomial of $M(\mathcal{A})$ from the arrangement \mathcal{A} .

Corollary 4.3.4. Let \mathcal{A} be a (locally) finite hyperplane arrangement in \mathbb{R}^n , and let $x_0 \in \mathbb{R}^n$ be a generic point. The k -th Betti number of the complement $M(\mathcal{A})$ is equal to the number of chambers C such that the projection of x_0 onto C lies in the relative interior of a face F_C of codimension k . Equivalently, the Poincaré polynomial of \mathcal{A} is given by

$$\pi(\mathcal{A}, t) = \sum_{C \in \mathcal{C}(\mathcal{A})} t^{\text{codim } F_C}.$$

In Section 4.4 we use Euclidean matchings to obtain a proof of Brieskorn’s Lemma (for locally finite complexified arrangements) which makes no use of algebraic geometry. In addition, we show that for every flat X there exist Euclidean matchings on $\text{Sal}(\mathcal{A})$ for which the Morse complex of the subarrangement \mathcal{A}_X is naturally included into the Morse complex of \mathcal{A} .

Finally, in Section 4.5 we give an explicit description of the algebraic Morse complex that computes the homology of $M(\mathcal{A})$ with coefficients in an abelian local system, for any locally finite line arrangement \mathcal{A} in \mathbb{R}^2 . We compare our result with the one of Gaiffi and Salvetti [GS09], where similar formulas are obtained in the case of finite line arrangements (using the polar matchings of Salvetti and Settepanella [SS07]).

4.2 EUCLIDEAN MATCHINGS

In this section we are going to construct a valid order \dashv_{eu} on the set of chambers \mathcal{C} , for any locally finite arrangement \mathcal{A} , using the Euclidean distance d in \mathbb{R}^n .

If K is a closed convex subset of \mathbb{R}^n , denote by $\rho_K(x)$ the projection of a point $x \in \mathbb{R}^n$ onto K . The point $\rho_K(x)$ is the unique point $y \in K$ such that $d(x, y) = d(x, K)$.

The first step is to prove that there exist a lot of *generic points* with respect to the arrangement \mathcal{A} . For this, we need the following technical lemma. By *measure* we always mean the Lebesgue measure in \mathbb{R}^n .

Lemma 4.2.1. Let K_1 and K_2 be two closed convex subsets of \mathbb{R}^n . Let

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid d(x, K_1) = d(x, K_2) \text{ and } \rho_{K_1}(x) \neq \rho_{K_2}(x)\}.$$

Then \mathcal{S} has measure zero.

Proof. This proof was suggested by Federico Glaudo. Let $d_i(x) = d(x, K_i)$ for $i = 1, 2$. Each function $d_i: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable on $\mathbb{R}^n \setminus K_i$ by [GM12, Lemma 2.19], and its gradient in a point $x \notin K_i$ is the versor with direction $x - \rho_{K_i}(x)$.

Let $f(x) = d_1(x) - d_2(x)$. Denote by A the open set of points $x \in \mathbb{R}^n \setminus (K_1 \cup K_2)$ such that $\rho_{K_1}(x) \neq \rho_{K_2}(x)$. On this set, the function f is differentiable and its gradient does not vanish. It is known that the gradient of f must vanish almost everywhere on $A \cap f^{-1}(0)$ [EG92, Corollary 1 of Section 3.1], hence $A \cap f^{-1}(0)$ has measure zero.

It is easy to check that the points in $K_1 \cup K_2$ cannot belong to \mathcal{S} . Then $\mathcal{S} = A \cap f^{-1}(0)$ has measure zero. \square

Lemma 4.2.2 (Generic points). Given a locally finite hyperplane arrangement \mathcal{A} in \mathbb{R}^n , let $\mathcal{G} \subseteq \mathbb{R}^n$ be the set of points $x \in \mathbb{R}^n$ such that:

- (i) for every $C, C' \in \mathcal{C}$ with $d(x, C) = d(x, C')$, we have $\rho_C(x) = \rho_{C'}(x) \in C \cap C'$;
- (ii) for every $L, L' \in \mathcal{L}$ with $L' \subsetneq L$, we have $d(x, L') > d(x, L)$.

Then the complement of \mathcal{G} has measure zero. In particular, \mathcal{G} is dense in \mathbb{R}^n .

Proof. Given $C, C' \in \mathcal{C}$, denote by $\mathcal{S}_{C,C'}$ the set of points $x \in \mathbb{R}^n$ such that $d(x, C) = d(x, C')$ and $\rho_C(x) \neq \rho_{C'}(x)$. By Lemma 4.2.1, every $\mathcal{S}_{C,C'}$ has measure zero.

Similarly, for every $L, L' \in \mathcal{L}$ with $L' \subsetneq L$, denote by $\mathcal{T}_{L,L'}$ the set of points $x \in \mathbb{R}^n$ such that $d(x, L') = d(x, L)$. We have that $\mathcal{T}_{L,L'}$ is an affine subspace of \mathbb{R}^n of codimension at least 1, and in particular it has measure zero.

The complement of \mathcal{G} is the union of all the sets $\mathcal{S}_{C,C'}$ for $C, C' \in \mathcal{C}$ and $\mathcal{T}_{L,L'}$ for $L, L' \in \mathcal{L}$ with $L' \subsetneq L$. This is a finite or countable union of sets of measure zero, hence it has measure zero. \square

We call *generic points* the elements of \mathcal{G} , as defined in Lemma 4.2.2. Notice that, by condition (ii) with $L = \mathbb{R}^n$, a generic point must lie in the complement of \mathcal{A} .

Remark 4.2.3. An equivalent definition of a generic point is the following: $x_0 \in \mathbb{R}^n$ is generic with respect to \mathcal{A} if and only if every flat of \mathcal{A} has a different distance from x_0 . Indeed, this definition immediately implies condition (ii) of Lemma 4.2.2. It also implies condition (i), because for any chamber C we have $d(x_0, C) = d(x_0, L)$, where L is the smallest flat that contains $\rho_C(x_0)$. Conversely, suppose that x_0 satisfies both conditions (i) and (ii). Given two flats $L, L' \in \mathcal{L}$ with $d(x_0, L) = d(x_0, L')$, by condition (ii) the projections $\rho_L(x_0)$ and $\rho_{L'}(x_0)$ must lie in the relative interior of faces $F, F' \in \mathcal{F}$ with $|F| = L$ and $|F'| = L'$. Defining C as the chamber containing F and with the greatest distance from x_0 , we immediately obtain that $\rho_L(x_0) = \rho_C(x_0)$. If C' is defined in the same way (using F' and L'), the chambers C and C' violate condition (i) unless $L = L'$. With this equivalent definition, it is possible to prove Lemma 4.2.2 in an alternative way without using Lemma 4.2.1 (cf. Lemma 4.3.2).

We are now able to define Euclidean orders.

Definition 4.2.4 (Euclidean orders). A total order \preceq_{eu} on the set of chambers \mathcal{C} is *Euclidean* if there exists a generic point x_0 such that $C \preceq_{\text{eu}} C'$ implies $d(x_0, C) \leq d(x_0, C')$. The point x_0 is called a *base point* of the Euclidean order \preceq_{eu} .

In other words, a Euclidean order is a linear extension of the partial order on \mathcal{C} given by $C < C'$ if $d(x_0, C) < d(x_0, C')$, for some fixed generic point $x_0 \in \mathbb{R}^n$. In particular, for every generic point x_0 there exists at least one Euclidean order with x_0 as a base point. Since the set of generic points is dense, we immediately get the following corollary.

Corollary 4.2.5. For every chamber $C_0 \in \mathcal{C}$, there exists a Euclidean order \preceq_{eu} that starts with C_0 .

Proof. Take the base point x_0 in the interior of the chamber C_0 . □

See Figure 4.1 for an example of a Euclidean order. We now prove that every Euclidean order is valid, in the sense of Definition 3.2.6.

Theorem 4.2.6. Let \preceq_{eu} be a Euclidean order with base point x_0 . For every chamber C , let $x_C = \rho_C(x_0)$, and let F_C be the smallest face of C that contains x_C . Then $\mathcal{J}(C)$ is the principal upper ideal generated by $X_C = |F_C|$. Therefore \preceq_{eu} is a valid order.

Proof. First we want to prove that $X_C \in \mathcal{J}(C)$. This is equivalent to proving that for every chamber $C' \preceq_{\text{eu}} C$ there exists a hyperplane $H \in \text{supp}(X_C) \cap s(C, C')$. We have that $\rho_{x_C}(x_0) = x_C$ because F_C is the smallest face that contains x_C . Then it is also true that $\rho_{\pi_{x_C}(C)}(x_0) = x_C$. Given a chamber $C' \preceq_{\text{eu}} C$, we have two possibilities.

- $d(x_0, C') < d(x_0, C)$. Then $C' \not\subseteq \pi_{x_C}(C)$, because all the points of $\pi_{x_C}(C)$ have distance at least $d(x_0, C)$ from x_0 . This means that there exists a hyperplane $H \in \text{supp}(X_C) = \mathcal{A}_{x_C}$ which separates C and C' .
- $d(x_0, C') = d(x_0, C)$. Since x_0 is a generic point, we have that $x_C = x_{C'} \in C \cap C'$. Then F_C is a common face of C and C' , and every hyperplane in $s(C, C')$ contains F_C .

Now we want to prove that $X \subseteq X_C$ for every $X \in \mathcal{J}(C)$. Suppose by contradiction that $X \not\subseteq X_C$ for some $X \in \mathcal{J}(C)$. In particular, we have $X_C \neq \mathbb{R}^n$ and thus $x_0 \neq x_C$. We first prove that $\text{supp}(X_C \cup X)$ is non-empty.

Let C' be the chamber of \mathcal{A} such that $x_0 \in \pi_{x_C}(C')$ and $C' \prec F_C$. Since $x_C \in X_C \subseteq \pi_{x_C}(C')$, the entire line segment ℓ from x_0 to x_C is contained in $\pi_{x_C}(C')$. Then there is a neighbourhood of x_C in ℓ which is contained in C' , hence $d(x_0, C') < d(x_0, x_C)$ and therefore $C' \preceq_{\text{eu}} C$. Since $X \in \mathcal{J}(C)$, there exists a hyperplane $H \in \text{supp}(X) \cap s(C, C')$. We also have that $F_C \subseteq C \cap C'$, and thus $X_C \subseteq H$.

Consider now the flat $X' = \bigcap \{Z \in \mathcal{L} \mid X_C \cup X \subseteq Z\}$, i.e. the meet of X_C and X in \mathcal{L} . The flat X' is contained in the hyperplane H constructed above, so in particular $X' \neq \mathbb{R}^n$. In addition, since $X \not\subseteq X_C$, X' is different from X_C . Then the point $y_0 = \rho_{X'}(x_0)$ is different from x_C , and we have $d(x_0, y_0) < d(x_0, x_C)$, because x_0 is generic (see condition (ii) of Lemma 4.2.2). Let F be the smallest face that contains the line segment $[x_C, x_C + \epsilon(y_0 - x_C)]$ for some $\epsilon > 0$. By construction, for every chamber C'' such that $C'' \preceq F$ we have that $C'' \dashv_{\text{eu}} C$. This holds in particular for $C'' = C.F$. Then we have $\text{supp}(F) \cap s(C, C'') = \emptyset$.

Since $X \in \mathcal{J}(C)$ and $C'' \dashv_{\text{eu}} C$, there exists a hyperplane $H \in \text{supp}(X) \cap s(C, C'')$. By construction, $x_C \in C \cap C''$ and then X_C is contained in every hyperplane of $s(C, C'')$. In particular, $X_C \subseteq H$. Therefore $X_C \cup X \subseteq H$, which means that $H \in \text{supp}(X_C \cup X) \subseteq \text{supp}(X')$. Both x_C and y_0 belong to X' , hence $F \subseteq X'$. Putting everything together, we get $H \in \text{supp}(X') \cap s(C, C'') \subseteq \text{supp}(F) \cap s(C, C'') = \emptyset$. This is a contradiction. \square

Since Euclidean orders are valid, we are able to construct acyclic matchings on the Salvetti complex of any arrangement.

Definition 4.2.7 (Euclidean matching). Let \mathcal{A} be a locally finite hyperplane arrangement in \mathbb{R}^n . We say that an acyclic matching \mathcal{M} on $\text{Sal}(\mathcal{A})$ is a *Euclidean matching* with base point $x_0 \in \mathbb{R}^n$ if:

- (i) the point x_0 is generic with respect to \mathcal{A} ;
- (ii) \mathcal{M} is homogeneous with respect to the poset map $\eta: \text{Sal}(\mathcal{A}) \rightarrow \mathcal{C} \times \mathcal{C}$ induced by a Euclidean order \dashv_{eu} with base point x_0 (defined as in the proof of Theorem 3.3.10);
- (iii) there is exactly one critical cell $\langle C, F_C \rangle$ for every chamber $C \in \mathcal{C}$, where F_C is the smallest face of C that contains $\rho_C(x_0)$.

Notice that, by condition (ii), a Euclidean matching is also proper.

Theorem 4.2.8. Let \mathcal{A} be a locally finite arrangement in \mathbb{R}^n . For every generic point $x_0 \in \mathbb{R}^n$, there exists a Euclidean matching on $\text{Sal}(\mathcal{A})$ with base point x_0 .

Proof. It follows from Theorems 3.3.10 and 4.2.6. \square

Remark 4.2.9. For a given generic point x_0 , there might be more than one Euclidean order \dashv_{eu} with base point x_0 . Nonetheless, all Euclidean orders with a given base point produce the same faces F_C (by Theorem 4.2.6) and the same critical cells (by Theorem 3.3.10). The decomposition

$$\text{Sal}(\mathcal{A}) = \bigsqcup_{C \in \mathcal{C}} N(C)$$

also depends only on x_0 (by Lemma 3.3.1), and therefore the definition of a Euclidean matching is not influenced by the choice of \neg_{eu} (once the base point x_0 is fixed).

4.3 MINIMALITY

In this section we prove that a Euclidean matching yields a minimal Morse complex. In order to do so, we first prove two lemmas about generic points.

Lemma 4.3.1. Let $x_0 \in \mathbb{R}^n$. If x_0 is generic with respect to an arrangement \mathcal{A} , then it is also generic with respect to any subarrangement $\mathcal{A}' \subseteq \mathcal{A}$.

Proof. Condition (i) for \mathcal{A}' holds because a chamber of \mathcal{A}' is a union of chambers of \mathcal{A} . Condition (ii) follows from the fact that $\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$. \square

Lemma 4.3.2. Let \mathcal{A} be a locally finite arrangement in \mathbb{R}^n , and let $x_0 \in \mathbb{R}^n$ be a generic point with respect to \mathcal{A} . Let $\mathcal{H} \subseteq (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$ be the set of elements (a_1, \dots, a_n, b) such that x_0 is generic also with respect to the arrangement $\mathcal{A} \cup \{H\}$, where H is the hyperplane defined by the equation $a_1x_1 + \dots + a_nx_n = b$. Then the complement of \mathcal{H} in \mathbb{R}^{n+1} has measure zero. In particular, \mathcal{H} is dense in \mathbb{R}^{n+1} .

Proof. In this proof we use the equivalent definition of a generic point given in Remark 4.2.3. Assume that H intersects generically every flat $X \in \mathcal{L}(\mathcal{A})$, i.e. $\text{codim}(X \cap H) = \text{codim}(X) + 1$. This condition excludes a subset of measure zero in \mathbb{R}^{n+1} .

Since x_0 is generic with respect to \mathcal{A} , the distances between x_0 and the flats of \mathcal{A} are all distinct. Consider now a flat of $\mathcal{A} \cup \{H\}$ of the form $X \cap H$, for some flat $X \in \mathcal{L}(\mathcal{A})$ of dimension ≥ 1 . The squared distance $d^2(x_0, X \cap H)$ is a rational function of the coefficients (a_1, \dots, a_n, b) that define H .

Given two flats $X, Y \in \mathcal{L}(\mathcal{A})$ with $\dim(X) \geq 1$, the condition $d^2(x_0, X \cap H) = d^2(x_0, Y)$ can be written as a polynomial equation $p(a_1, \dots, a_n, b) = 0$. This equation is not satisfied if $d(x_0, H) > d(x_0, Y)$, therefore the polynomial p is not identically zero. Then the zero locus of p has measure zero.

Similarly, given two flats $X, Y \in \mathcal{L}(\mathcal{A})$ with $\dim(X) \geq 1$ and $\dim(Y) \geq 1$, the condition $d^2(x_0, X \cap H) = d^2(x_0, Y \cap H)$ can be written as a polynomial equation $q(a_1, \dots, a_n, b) = 0$. Up to exchanging X and Y , we can assume that $\rho_X(x_0) \notin Y$, because $d(x_0, X) \neq d(x_0, Y)$. If H is the hyperplane orthogonal to the vector $\rho_X(x_0) - x_0$ that passes through $\rho_X(x_0)$, we have $d(x_0, X \cap H) = d(x_0, X)$ and $d(x_0, Y \cap H) > d(x_0, H) = d(x_0, X)$ (the inequality is strict because Y does not contain $\rho_H(x_0) = \rho_X(x_0)$). Therefore the polynomial q is not identically zero, and the zero locus of q has measure zero.

Then the complement of \mathcal{H} is contained in a finite or countable union of sets of measure zero, thus it has measure zero. \square

Theorem 4.3.3 (Minimality). Let \mathcal{A} be a locally finite hyperplane arrangement in \mathbb{R}^n , and let \mathcal{M} be a Euclidean matching on $\text{Sal}(\mathcal{A})$ with base point x_0 . Then the associated Morse complex $\text{Sal}(\mathcal{A})_{\mathcal{M}}$ is minimal (i.e. all the incidence numbers vanish).

Proof. If the arrangement \mathcal{A} is finite, it is well known that the sum of the Betti numbers of $\text{Sal}(\mathcal{A})$ is equal to the number of chambers [OS80, Zas97]. By Theorem 3.3.10, the critical cells of \mathcal{M} are in bijection with the chambers. Then the Morse complex is minimal.

Suppose from now on that \mathcal{A} is infinite. Fix a chamber $C \in \mathcal{C}$, and consider the associated critical cell $\langle C, F_C \rangle \in N(C)$. Recall from the proof of Theorem 3.3.10 the definition of the poset map $\eta: \text{Sal}(\mathcal{A}) \rightarrow \mathcal{C} \times \mathcal{C}$, and let $(C, E) = \eta(\langle C, F_C \rangle)$. Since the matching is proper, the set $\eta^{-1}((\mathcal{C} \times \mathcal{C})_{\leq (C, E)})$ is finite.

Consider now the finite set of faces

$$\mathcal{U} = \{F \in \mathcal{F} \mid \langle D, F \rangle \in \eta^{-1}((\mathcal{C} \times \mathcal{C})_{\leq (C, E)}) \text{ for some chamber } D \in \mathcal{C}\}.$$

Let $B \subseteq \mathbb{R}^n$ be an open Euclidean ball centered in x_0 that contains the projection $\rho_F(x_0)$ for every face $F \in \mathcal{U}$. Let $\tilde{\mathcal{A}}$ be a set of $n + 1$ hyperplanes that do not intersect B , such that: x_0 is still generic with respect to $\mathcal{A} \cup \tilde{\mathcal{A}}$; the chamber K of the arrangement $\tilde{\mathcal{A}}$ containing B is bounded. Such an arrangement $\tilde{\mathcal{A}}$ exists thanks to Lemma 4.3.2. Consider the finite arrangement

$$\mathcal{A}' = \{H \in \mathcal{A} \mid H \cap K \neq \emptyset\} \cup \tilde{\mathcal{A}},$$

and let $\mathcal{F}_K \subseteq \mathcal{F}(\mathcal{A})$ be the set of faces of \mathcal{A} that intersect the interior of K . Notice that, by construction, we have $\mathcal{U} \subseteq \mathcal{F}_K$. In addition, there is a natural order-preserving and rank-preserving injection $\varphi: \mathcal{F}_K \rightarrow \mathcal{F}(\mathcal{A}')$ given by $\varphi(F) = F \cap K$. The image of φ consists of the faces of \mathcal{A}' that intersect the interior of K .

By Lemma 4.3.1, x_0 is still generic with respect to \mathcal{A}' , and all the chambers $D \in \mathcal{C}(\mathcal{A})$ with $d(x_0, D) \leq d(x_0, C)$ intersect the interior of K . Then, given a Euclidean order \dashv_{eu} on $\mathcal{C}(\mathcal{A})$ with base point x_0 , there exists a Euclidean order \dashv'_{eu} on $\mathcal{C}(\mathcal{A}')$ with base point x_0 such that φ is an order-preserving bijection between the initial segment of $(\mathcal{C}(\mathcal{A}), \dashv_{\text{eu}})$ up to C and the initial segment of $(\mathcal{C}(\mathcal{A}'), \dashv'_{\text{eu}})$ up to $\varphi(C)$.

Consider the subcomplex $S = \eta^{-1}((\mathcal{C} \times \mathcal{C})_{\leq (C, E)})$ of $\text{Sal}(\mathcal{A})$. Since $\mathcal{U} \subseteq \mathcal{F}_K$, the map φ induces an order-preserving and orientation-preserving injection $\psi: S \rightarrow \text{Sal}(\mathcal{A}')$ that maps a cell $\langle D, G \rangle \in S$ to the cell $\langle \varphi(D), \varphi(G) \rangle \in \text{Sal}(\mathcal{A}')$. Let $S' = \psi(S)$ be the copy of S inside $\text{Sal}(\mathcal{A}')$. By definition of S , a fiber of η is either disjoint from S or entirely contained in S . Therefore, a non-critical cell of S is matched with another cell of S .

We now use the order \dashv'_{eu} to construct a Euclidean matching \mathcal{M}' on $\text{Sal}(\mathcal{A}')$. Denote by $\eta': \text{Sal}(\mathcal{A}') \rightarrow \mathcal{C}(\mathcal{A}') \times \mathcal{C}(\mathcal{A}')$ the analogue of η for the

arrangement \mathcal{A}' (see the proof of Theorem 3.3.10). Consider a fiber $\eta'^{-1}(C', E')$ that intersects S' . Then there is some cell $\langle D', G' \rangle \in \eta'^{-1}(C', E') \cap S'$, with $\langle D', G' \rangle = \psi(\langle D, G \rangle)$ for some $\langle D, G \rangle \in S$. If we define $(\bar{C}, \bar{E}) = \eta(\langle D, G \rangle)$, we have that $\varphi(\bar{C}) = C'$ and $\varphi(\bar{E}) = E'$, because by construction the cell $\langle \varphi(D), \varphi(G) \rangle$ is in the fiber $\eta^{-1}(\varphi(\bar{C}), \varphi(\bar{E}))$. By Corollary 3.3.5 and Lemma 3.3.7, the fiber $\eta'^{-1}(C', E')$ is isomorphic to the poset of faces of $E' \cap X_{C'}$ visible from some point $y_{C'}$ in the relative interior of $C' \cap X_{C'}$. By construction of \mathcal{A}' , the map φ induces a bijection between the faces of $E' \cap X_{C'}$ visible from $y_{C'}$ and the faces of $\bar{E} \cap X_{\bar{C}} = \bar{E} \cap X_{C'}$ visible from $y_{C'}$: if F is a visible face of $\bar{E} \cap X_{C'}$, then $F \in \mathcal{U}$ and so $\varphi(F)$ is still visible; conversely, a visible face F' of $E' \cap X_{C'}$ cannot be contained in any hyperplane of $\bar{\mathcal{A}}$, and by construction of \mathcal{A}' it must be also a face of \bar{E} . Therefore the fiber $\eta'^{-1}(C', E')$ is the isomorphic image of the fiber $\eta^{-1}(\bar{C}, \bar{E})$ under the map ψ .

We have proved that a fiber of η' is either disjoint from S' or entirely contained in S' . Then we can choose the Euclidean matching \mathcal{M}' so that its restriction to S' coincides with the image of the restriction of \mathcal{M} to S under the isomorphism $\psi: S \rightarrow S'$. In particular, a cell $\langle D, G \rangle \in S$ is \mathcal{M} -critical if and only if $\psi(\langle D, G \rangle) \in S'$ is \mathcal{M}' -critical.

Consider now a \mathcal{M} -critical cell $\langle D, G \rangle \in \text{Sal}(\mathcal{A})$ such that there is at least one alternating path from $\langle C, F_C \rangle$ to $\langle D, G \rangle$. Since \mathcal{M} is homogeneous with respect to η , every alternating path starting from $\langle C, F \rangle$ is entirely contained in S . In particular, $\langle D, G \rangle \in S$. Then the map $\psi: S \rightarrow S'$ induces a bijection between the alternating paths from $\langle C, F \rangle$ to $\langle D, G \rangle$ in $\text{Sal}(\mathcal{A})$ (with respect to the matching \mathcal{M}) and the alternating paths from $\psi(\langle C, F \rangle)$ to $\psi(\langle D, G \rangle)$ in $\text{Sal}(\mathcal{A}')$ (with respect to the matching \mathcal{M}'). In particular, the incidence number between $\langle C, F \rangle$ and $\langle D, G \rangle$ in the Morse complex $\text{Sal}(\mathcal{A})_{\mathcal{M}}$ is the same as the incidence number between $\psi(\langle C, F \rangle)$ and $\psi(\langle D, G \rangle)$ in the Morse complex $\text{Sal}(\mathcal{A}')_{\mathcal{M}'}$. Since \mathcal{A}' is finite, the Morse complex $\text{Sal}(\mathcal{A}')_{\mathcal{M}'}$ is minimal and all its incidence numbers vanish. Therefore the incidence number between $\langle C, F \rangle$ and $\langle D, G \rangle$ in $\text{Sal}(\mathcal{A})_{\mathcal{M}}$ also vanishes. \square

The following result is a direct consequence of Theorems 4.2.8 and 4.3.3. It gives a simple geometric way to compute the Betti numbers of the complement of an arrangement.

Corollary 4.3.4 (Betti numbers). Let \mathcal{A} be a (locally) finite hyperplane arrangement in \mathbb{R}^n , and let $x_0 \in \mathbb{R}^n$ be a generic point. The k -th Betti number of the complement $M(\mathcal{A})$ is equal to the number of chambers C such that the projection $\rho_C(x_0)$ lies in the relative interior of a face F_C of codimension k . Equivalently, the Poincaré polynomial of \mathcal{A} is given by

$$\pi(\mathcal{A}, t) = \sum_{C \in \mathcal{C}(\mathcal{A})} t^{\text{codim } F_C}. \quad \square$$

Example 4.3.5. Consider the line arrangement \mathcal{A} of Figure 4.1. For the given generic point x_0 in the interior of C_0 , the computation of the Betti numbers b_i

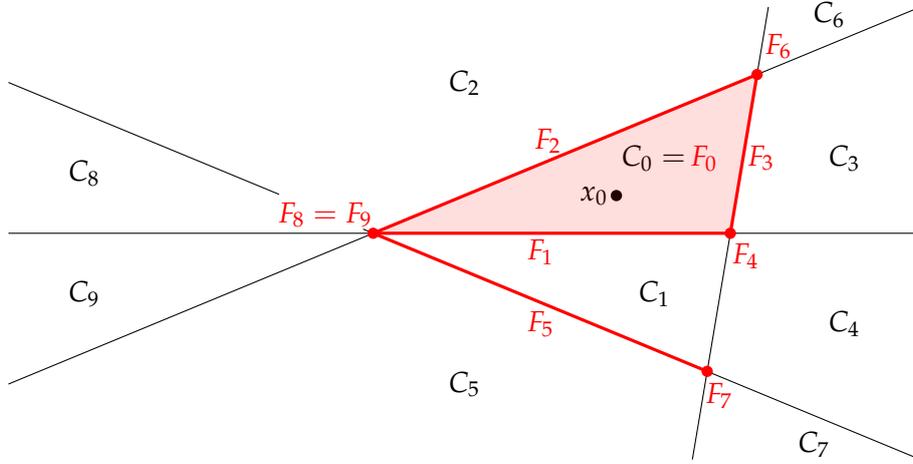


Figure 4.1: Euclidean order with respect to x_0 . The faces $F_i = F_{C_i}$ defined in Theorem 4.2.6 are highlighted.

according to Corollary 4.3.4 goes as follows: there is one chamber (namely C_0) such that the projection of x_0 lies in its interior, so $b_0 = 1$; there are four chambers (namely C_1, C_2, C_3 and C_5) such that the projection of x_0 lies in the interior of a 1-dimensional face, so $b_1 = 4$; finally, for the remaining chambers (C_4, C_6, C_7, C_8 and C_9) the projection of x_0 is a 0-dimensional face, so $b_2 = 5$.

Remark 4.3.6. For any choice of the generic point x_0 , the only chamber that contributes to the 0-th Betti number is the one containing x_0 . In addition, for every hyperplane $H \in \mathcal{A}$ there is exactly one chamber C such that $\rho_C(x_0) \in H$ and $\rho_C(x_0) \notin H'$ for every $H' \in \mathcal{A} \setminus \{H\}$. Therefore Corollary 4.3.4 immediately implies the well-known facts that $b_0(\mathcal{A}) = 1$ and $b_1(\mathcal{A}) = |\mathcal{A}|$.

4.4 BRIESKORN'S LEMMA AND NATURALITY

In this section we are going to relate the Morse complex of \mathcal{A} , constructed using a Euclidean matching, to the Morse complexes of subarrangements \mathcal{A}_X .

Given a flat $X \in \mathcal{L}(\mathcal{A})$, for every face $\bar{F} \in \mathcal{F}(\mathcal{A})$ such that $|\bar{F}| = X$ there is a natural inclusion of $\text{Sal}(\mathcal{A}_X)$ into $\text{Sal}(\mathcal{A})$. It maps a cell $\langle D, G \rangle \in \text{Sal}(\mathcal{A}_X)$ to the unique cell $\langle C, F \rangle \in \text{Sal}(\mathcal{A})$ such that $\bar{F} \subseteq F \subseteq G$, $\dim F = \dim G$, and $C \subseteq D$. We call this the inclusion of $\text{Sal}(\mathcal{A}_X)$ into $\text{Sal}(\mathcal{A})$ around \bar{F} . Geometrically, this corresponds to including the complement of \mathcal{A}_X^C , intersected with a neighbourhood of some point in the interior of \bar{F} , into $M(\mathcal{A})$. The inclusions $\text{Sal}(\mathcal{A}_X) \hookrightarrow \text{Sal}(\mathcal{A})$ that we are going to consider in this section are always of this type, for some face $\bar{F} \in \mathcal{F}(\mathcal{A})$ with $|\bar{F}| = X$.

Recall from Definition 4.2.7 that a Euclidean matching has a critical cell $\langle C, F_C \rangle \in \text{Sal}(\mathcal{A})$ for every chamber C , where F_C is the smallest face of C

containing $\rho_C(x_0)$. Every critical cell $\langle C, F_C \rangle$ is thus associated to a flat $X_C = |F_C|$. Conversely, given a flat $X \in \mathcal{L}(\mathcal{A})$, the critical cells $\langle C, F_C \rangle$ associated to X are exactly those for which $\rho_C(x_0) = \rho_X(x_0)$.

This simple observation yields a proof of Brieskorn's Lemma, a classical result in the theory of hyperplane arrangements due to Brieskorn [Bri73]. See also [OT13, Lemma 5.91] and [CD17, Proposition 3.3.3].

Lemma 4.4.1 (Brieskorn's Lemma [Bri73]). Let \mathcal{A} be a locally finite arrangement in \mathbb{R}^n . For every $k \geq 0$, there is an isomorphism

$$\theta_k: \bigoplus_{X \in \mathcal{L}_k} H_k(M(\mathcal{A}_X); \mathbb{Z}) \rightarrow H_k(M(\mathcal{A}); \mathbb{Z})$$

induced by suitable inclusions $j_X: \text{Sal}(\mathcal{A}_X) \hookrightarrow \text{Sal}(\mathcal{A})$ of CW complexes. The inverse isomorphism θ_k^{-1} is induced by the natural inclusion maps $i_X: M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$.

Proof. Let $x_0 \in \mathbb{R}^n$ be a generic point with respect to \mathcal{A} , and let \mathcal{M} be a Euclidean matching on $\text{Sal}(\mathcal{A})$ with base point x_0 . Let $X \in \mathcal{L}_k$ be a flat of codimension k . By Lemma 4.3.1, the point x_0 is generic also with respect to the subarrangement \mathcal{A}_X . Consider the inclusion $j_X: \text{Sal}(\mathcal{A}_X) \hookrightarrow \text{Sal}(\mathcal{A})$ around the unique face of \mathcal{A} containing the projection $\rho_X(x_0)$. Let \mathcal{M}_X be a Euclidean matching on $\text{Sal}(\mathcal{A}_X)$ with base point x_0 .

All homology groups in this proof are with integer coefficients. By Theorem 4.3.3, we have that $H_k(\text{Sal}(\mathcal{A}))$ is a free abelian group generated by elements of the form

$$\langle C, F_C \rangle + \text{a finite sum of non-critical } k\text{-cells}$$

for each critical k -cell $\langle C, F_C \rangle$ of $\text{Sal}(\mathcal{A})$. Similarly, for every flat $X \in \mathcal{L}_k$, we have that $H_k(\text{Sal}(\mathcal{A}_X))$ is a free abelian group generated by elements of the same form as above, one for every critical k -cell of $\text{Sal}(\mathcal{A}_X)$. The critical k -cells of $\text{Sal}(\mathcal{A}_X)$ are in bijection (through the map j_X) with the critical k -cells $\langle C, F \rangle$ of $\text{Sal}(\mathcal{A})$ such that $|F| = X$. Then the inclusions j_X induce an isomorphism

$$\bar{\theta}_k: \bigoplus_{X \in \mathcal{L}_k} H_k(\text{Sal}(\mathcal{A}_X)) \rightarrow H_k(\text{Sal}(\mathcal{A})).$$

Let $\varphi: \text{Sal}(\mathcal{A}) \xrightarrow{\simeq} M(\mathcal{A})$ and $\varphi_X: \text{Sal}(\mathcal{A}_X) \xrightarrow{\simeq} M(\mathcal{A}_X)$ be the homotopy equivalences constructed in [Sal87]. Then the composition

$$\bigoplus_{X \in \mathcal{L}_k} H_k(M(\mathcal{A}_X)) \xrightarrow{\bigoplus (\varphi_X)^{-1}} \bigoplus_{X \in \mathcal{L}_k} H_k(\text{Sal}(\mathcal{A}_X)) \xrightarrow{\bar{\theta}_k} H_k(\text{Sal}(\mathcal{A})) \xrightarrow{\varphi_*} H_k(M(\mathcal{A}))$$

is the isomorphism θ_k of the statement.

By naturality of Salvetti's construction, the following diagram is commutative up to homotopy.

$$\begin{array}{ccc}
 \text{Sal}(\mathcal{A}_X) & \xleftarrow{j_X} & \text{Sal}(\mathcal{A}) \\
 \downarrow \varphi_X & & \downarrow \varphi \\
 M(\mathcal{A}_X) & \xleftarrow{i_X} & M(\mathcal{A})
 \end{array}$$

Looking at the induced commutative diagram in homology, we obtain that the inverse isomorphism θ_k^{-1} is induced by the inclusion maps i_X . \square

Remark 4.4.2. With an idea similar to the one employed in the previous proof, it might be possible to derive a version of Brieskorn's Lemma for abstract oriented matroids. We leave this as a potential direction for future work.

If we fix a flat $X \in \mathcal{L}(\mathcal{A})$, it is possible to choose the base point x_0 so that the Morse complex of $\text{Sal}(\mathcal{A}_X)$ injects into the Morse complex of $\text{Sal}(\mathcal{A})$. We prove this naturality property in the following lemma.

Lemma 4.4.3. Let $X \in \mathcal{L}(\mathcal{A})$ be a flat, and fix an inclusion $j: \text{Sal}(\mathcal{A}_X) \hookrightarrow \text{Sal}(\mathcal{A})$ around some face \bar{F} with $|\bar{F}| = X$. There exist Euclidean matchings \mathcal{M}_X and \mathcal{M} , on $\text{Sal}(\mathcal{A}_X)$ and $\text{Sal}(\mathcal{A})$ respectively, such that:

- (i) they share the same base point x_0 ;
- (ii) $j(\mathcal{M}_X) \subseteq \mathcal{M}$;
- (iii) the inclusion j induces an inclusion of the Morse complex of $\text{Sal}(\mathcal{A}_X)$ into the Morse complex of $\text{Sal}(\mathcal{A})$.

Proof. Let $x_0 \in \mathbb{R}^n$ be a generic point such that $d(x_0, \bar{F}) < d(x_0, H)$ for every hyperplane $H \in \mathcal{A} \setminus \mathcal{A}_X$ (the existence of x_0 follows from Lemma 4.2.2). For example, we can choose a point y in the relative interior of \bar{F} , and then take x_0 in a small neighbourhood of y .

Let \preceq_{eu} and \preceq'_{eu} be Euclidean orders with base point x_0 on $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{A}_X)$, respectively. Notice that, by construction of x_0 , the total order \preceq_{eu} starts with the chambers containing \bar{F} .

Let $\eta: \text{Sal}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}) \times \mathcal{C}(\mathcal{A})$ be the poset map defined in the proof of Theorem 3.3.10, induced by the total order \preceq_{eu} . Let $\eta': \text{Sal}(\mathcal{A}_X) \rightarrow \mathcal{C}(\mathcal{A}_X) \times \mathcal{C}(\mathcal{A}_X)$ be the analogous poset map for the arrangement \mathcal{A}_X , induced by the total order \preceq'_{eu} . Then, for every pair of chambers $C, E \in \mathcal{C}(\mathcal{A})$ containing \bar{F} , we have

$$\eta^{-1}(C, E) = j(\eta'^{-1}(\pi_X(C), \pi_X(E))).$$

In other words, the inclusion j maps fibers of η' to fibers of η . Notice that, by Remark 4.2.9, these fibers only depend on x_0 and not on the particular choices of the Euclidean orders \preceq_{eu} and \preceq'_{eu} .

Let \mathcal{M}_X be a Euclidean matching on $\text{Sal}(\mathcal{A}_X)$ with base point x_0 . Recall that such a matching is constructed on the fibers of η (see Definition 4.2.7).

Then there exists a Euclidean matching \mathcal{M} on $\text{Sal}(\mathcal{A})$ with base point x_0 that contains $j(\mathcal{M}_X)$.

The alternating paths in $\text{Sal}(\mathcal{A})$ starting from cells in the subcomplex $j(\text{Sal}(\mathcal{A}_X))$ remain in this subcomplex. Therefore j induces an inclusion of the Morse complex of $\text{Sal}(\mathcal{A}_X)$ (with respect to the matching \mathcal{M}_X) into the Morse complex of $\text{Sal}(\mathcal{A})$ (with respect to the matching \mathcal{M}). \square

4.5 LOCAL HOMOLOGY OF LINE ARRANGEMENTS

Let \mathcal{A} be a locally finite line arrangement in \mathbb{R}^2 . In this section we are going to describe the algebraic Morse complex (associated to a Euclidean matching) that computes the homology of the complement $M(\mathcal{A})$ with coefficients in an abelian local system. Then we are going to compare the obtained complex with the algebraic complex of Gaiffi and Salvetti [GS09], which is based on the polar matching of Salvetti and Settepanella [SS07].

An abelian local system L on $M(\mathcal{A})$ is determined by the elements $t_\ell \in \text{Aut}(L)$ associated to elementary positive loops around every line $\ell \in \mathcal{A}$ [GS09, Section 2.4]. The boundaries ∂_i of the algebraic Morse complex are determined by the incidence numbers $[\langle D, G \rangle, \langle C, F \rangle]^{\mathcal{M}} \in \mathbb{Z}[t_\ell^{\pm 1}]_{\ell \in \mathcal{A}}$, between critical i -cells $\langle D, G \rangle$ and critical $(i-1)$ -cells $\langle C, F \rangle$, in the Morse complex.

We refer to [SS07, Section 5] for a detailed explanation of how to compute these incidence numbers, given an acyclic matching on the Salvetti complex $\text{Sal}(\mathcal{A})$. We only make the following substantial change of convention with respect to [SS07, GS09]: given a cell $\langle C, F \rangle$, we choose as its representative point the 0-cell $\langle C^F, C^F \rangle$, where C^F is the chamber opposite to C with respect to F (the role of the representative point is thoroughly described in [Ste43, Section 9]). It is more convenient to choose $\langle C^F, C^F \rangle$ instead of $\langle C, C \rangle$, because in this way two matched cells have the same representative point.

We recall some useful definitions and facts from [SS07, Chapter 5], adapting them to our different convention on the representative point. Given two chambers D and C , denote by $u(D, C)$ a combinatorial positive path of minimal length from $\langle D, D \rangle$ to $\langle C, C \rangle$, in the 1-skeleton of $\text{Sal}(\mathcal{A})$. In particular, let $\Gamma(C) = u(C, C_0)$ be a minimal positive path from the chamber C to a base chamber C_0 . Every path $u(D, C)$ crosses each hyperplane at most once by [SS07, Lemma 5.1]. Consider the closed path $\Gamma(D)^{-1} u(D, C) \Gamma(C)$, which starts from C_0 , passes through D and C , and then goes back to C_0 . This path determines an element $\bar{u}(D, C) \in H_1(M(\mathcal{A}))$ which is equal to the product of the positive loops around the hyperplanes in $s(C_0, C) \cap s(D, C)$. Then the incidence number $[\langle D, G \rangle, \langle C, F \rangle] \in \mathbb{Z}[t_\ell^{\pm 1}]_{\ell \in \mathcal{A}}$ between an i -cell $\langle D, G \rangle$ and an $(i-1)$ -cell $\langle C, F \rangle$ in $\text{Sal}(\mathcal{A})$ is given by

$$[\langle D, G \rangle : \langle C, F \rangle] = [\langle D, G \rangle : \langle C, F \rangle]_{\mathbb{Z}} \bar{u}(D^G, C^F),$$

where $[\langle D, G \rangle : \langle C, F \rangle]_{\mathbb{Z}} = \pm 1$ denotes the incidence number with integer coefficients.

Let $x_0 \in \mathbb{R}^2$ be a generic point with respect to the line arrangement \mathcal{A} , and fix a Euclidean matching \mathcal{M} on the Salvetti complex $\text{Sal}(\mathcal{A})$ with base point x_0 . Let C_0 be the chamber containing x_0 (this is the first chamber in any Euclidean order with base point x_0). Recall that the matching \mathcal{M} is constructed on the fibers of the map $\eta: \text{Sal}(\mathcal{A}) \rightarrow \mathcal{C} \times \mathcal{C}$.

To compute the algebraic Morse complex (see Section 1.3), we first need to describe the alternating paths between critical cells. The alternating paths between a critical 1-cell $\langle C, F \rangle$ and the only critical 0-cell $\langle C_0, C_0 \rangle$ are particularly simple, since all the 0-cells are in $N(C_0)$.

Lemma 4.5.1. Let $\langle C, F \rangle$ be a critical 1-cell. Denote by C' the unique chamber containing F other than C . There are exactly two alternating paths from $\langle C, F \rangle$ to the only critical 0-cell $\langle C_0, C_0 \rangle$:

- $\langle C, F \rangle \succ \langle C, C \rangle \prec \langle C', F \rangle \succ \langle C', C' \rangle \prec \cdots \succ \langle C_0, C_0 \rangle$
- $\langle C, F \rangle \succ \langle C', C' \rangle \prec \cdots \succ \langle C_0, C_0 \rangle$

(after $\langle C', C' \rangle$, they continue in the same way).

Proof. Since $\langle C, F \rangle$ is critical, the line $|F|$ separates C and C_0 . In the boundary of the 1-cell $\langle C, F \rangle$ there are the two 0-cells $\langle C, C \rangle$ and $\langle C', C' \rangle$. The 0-cell $\langle C, C \rangle$ is matched with the 1-cell $\langle C', F \rangle$, because these are the unique cells in the fiber $\eta^{-1}(C_0, C)$. Then an alternating path starting with $\langle C, F \rangle \succ \langle C, C \rangle$ is forced to continue with $\prec \langle C', F \rangle \succ \langle C', C' \rangle$. After $\langle C', C' \rangle$ there is exactly one way to continue the path, because every non-critical 0-cell is matched with some 1-cell, and this 1-cell has exactly one other 0-cell in its boundary. Since the matching is proper, one such path must eventually reach the critical 0-cell $\langle C_0, C_0 \rangle$. \square

We can use the previous lemma to compute the boundary ∂_1 . The resulting formula coincides with the one of [GS09, Proposition 4.1].

Proposition 4.5.2. The incidence number between a critical 1-cell $\langle C, F \rangle$ and the only critical 0-cell $\langle C_0, C_0 \rangle$ in the Morse complex is given by

$$[\langle C, F \rangle : \langle C_0, C_0 \rangle]^{\mathcal{M}} = (1 - t_{|F|}).$$

Proof. The orientation of a 1-cell $\langle \tilde{C}, \tilde{F} \rangle$ is defined so that $[\langle \tilde{C}, \tilde{F} \rangle, \langle \tilde{C}^{\tilde{F}}, \tilde{C}^{\tilde{F}} \rangle]_{\mathbb{Z}} = 1$. Now, if $\langle \tilde{C}, \tilde{F} \rangle \in N(C_0)$, then \tilde{C} is closer to C_0 with respect to $\tilde{C}^{\tilde{F}}$ and so we have that:

$$[\langle \tilde{C}, \tilde{F} \rangle, \langle \tilde{C}, \tilde{C} \rangle] = -1; \quad [[\langle \tilde{C}, \tilde{F} \rangle : \langle \tilde{C}^{\tilde{F}}, \tilde{C}^{\tilde{F}} \rangle] = 1.$$

By Lemma 4.5.1 we see that there are exactly two alternating paths between $\langle C, F \rangle$ and $\langle C_0, C_0 \rangle$, and by Theorem 1.3.1 the incidence number in the Morse complex is given by

$$[\langle C, F \rangle : \langle C_0, C_0 \rangle]^{\mathcal{M}} = [[\langle C, F \rangle : \langle C, C \rangle] + [\langle C, F \rangle : \langle C', C' \rangle].$$

Since $|F| \in s(C_0, C) \cap s(C^F, C)$, the first term is

$$[\langle C, F \rangle : \langle C, C \rangle] = [\langle C, F \rangle : \langle C, C \rangle]_{\mathbb{Z}} \bar{u}(C^F, C) = -t_{|F|},$$

The second term is given by

$$[\langle C, F \rangle : \langle C', C' \rangle] = [\langle C, F \rangle : \langle C', C' \rangle]_{\mathbb{Z}} \bar{u}(C^F, C') = \bar{u}(C', C') = 1. \quad \square$$

Now we want to compute the boundary ∂_2 . To simplify the notation, denote a 2-cell $\langle D, \{p\} \rangle$ also by $\langle D, p \rangle$, where $p \in \mathbb{R}^2$ is the intersection point of two or more lines of \mathcal{A} .

It is convenient to assign the orientation of the 2-cells so that they behave well with respect to the matching. Given a 2-cell $\langle D, p \rangle \notin N(C_0)$, we choose the orientation in the following way. Let ℓ, ℓ' be the two walls of D that pass through p . Let ℓ be the one that does not separate D from C_0 if it exists, or otherwise the closest one to x_0 . Then the orientation of $\langle D, p \rangle$ is the one for which $[\langle D, p \rangle, \langle D, \ell \rangle]_{\mathbb{Z}} = 1$. The orientation of the 2-cells in $N(C_0)$ is assigned arbitrarily. The reason of this choice is that the incidence number between two matched cells is always $+1$. Indeed, if C' is the chamber such that $X_{C'} = \ell'$, then $\langle D, p \rangle \in N(C')$ by construction.

We are going to show that there is a correspondence between alternating paths from critical 2-cells to critical 1-cells and certain sequences of elements of $\mathcal{L}_1(\mathcal{A})$. Consider an alternating path of the form

$$\langle D, p \rangle \succ \langle C_1, F_1 \rangle \prec \langle D_1, p_1 \rangle \succ \langle C_2, F_2 \rangle \prec \cdots \succ \langle C_n, F_n \rangle, \quad (4.1)$$

where $\langle D, p \rangle$ is a critical 2-cell and $\langle C_n, F_n \rangle$ is a critical 1-cell. By construction of the matching, none of the cells in (4.1) belongs to $N(C_0)$. We have that the starting cell $\langle D, p \rangle$ and the sequence (F_1, \dots, F_n) uniquely determine the alternating path. This is because for each i there are only two cells of the form $\langle C', F_i \rangle$ for some $C' \in \mathcal{C}$, and one of these cells is in $N(C_0)$. Then C_i is uniquely determined by F_i for every i , and $\langle D_i, p_i \rangle$ is the cell matched with $\langle C_i, F_i \rangle$.

We are now going to describe which sequences in $\mathcal{L}_1(\mathcal{A})$ give rise to an alternating path. Given a face $F \in \mathcal{L}_1(\mathcal{A})$, let $\ell = |F|$. If $\rho_\ell(x_0) \notin F$, we denote by $p(F)$ the endpoint of F which is closer to $\rho_\ell(x_0)$. In addition, let $C(F)$ be the unique chamber containing F such that $\langle C(F), F \rangle \notin N(C_0)$.

Definition 4.5.3. Given two different faces $F, G \in \mathcal{L}_1(\mathcal{A})$, we write $F \rightarrow G$ if

- $F \cap G = \{p(F)\}$;
- $|F| = |G|$, or F and C_0 lie in the same half-plane with respect to $|G|$.

Lemma 4.5.4. Let $\langle D, p \rangle$ be a critical 2-cell and $\langle C, F \rangle$ a critical 1-cell. The alternating paths between $\langle D, p \rangle$ and $\langle C, F \rangle$ are in one to one correspondence with the sequences in $\mathcal{L}_1(\mathcal{A})$ of the form $(F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_n = F)$ such that $\langle C(F_1), F_1 \rangle \prec \langle D, p \rangle$.

Proof. Consider an alternating path as in (4.1). We have already seen that such a path is completely determined by the starting cell $\langle D, p \rangle$ and by the sequence (F_1, \dots, F_n) . Clearly the condition $\langle C(F_1), F_1 \rangle \prec \langle D, p \rangle$ must be satisfied. We want to show that $F_i \rightarrow F_{i+1}$ for each $i = 1, \dots, n-1$.

Let E_i be the chamber opposite to $C(F_i)$ with respect to F_i . By construction of the matching, it is immediate to see that the cell $\langle C(F_i), F_i \rangle$ is matched with $\langle D(F_i), p(F_i) \rangle$, where $D(F_i)$ is the chamber opposite to E_i with respect to $p(F_i)$. By hypothesis we have that $\langle C(F_{i+1}), F_{i+1} \rangle \prec \langle D(F_i), p(F_i) \rangle$ which implies that $F_i \cap F_{i+1} = \{p(F_i)\}$ and that $D(F_i).F_{i+1} = C(F_{i+1})$. Since $\langle C(F_{i+1}), F_{i+1} \rangle \notin N(C_0)$, we have that C_0 and $C(F_{i+1})$ are in opposite half-planes with respect to $|F_{i+1}|$. The same is true for F_i and $C(F_{i+1})$, because $D(F_i)$ and F_i are in opposite half-planes with respect to $|F_{i+1}|$, unless $F_i \subset |F_{i+1}|$. Then we have that $F_i \rightarrow F_{i+1}$.

Conversely, we now prove that every sequence $(F_1 \rightarrow F_2 \rightarrow \dots \rightarrow F_n = F)$ satisfying $\langle C(F_1), F_1 \rangle \prec \langle D, p \rangle$ has an associated alternating path. We do this by induction on the length n of the sequence.

The case $n = 1$ is trivial, since we already know that $\langle C(F_1), F_1 \rangle \prec \langle D, p \rangle$. In the induction step, we need only to prove that $F \rightarrow G$ implies $\langle C(G), G \rangle \prec \langle D(F), p(F) \rangle$. From the first condition of Definition 4.5.3, we have that $G \prec \{p(F)\}$. We need to check that $D(F).G = C(G)$. By definition of $C(G)$, this is equivalent to proving that $D(F)$ and C_0 lie in opposite half-planes with respect to $|G|$. This is true because F and C_0 lie in the same half-plane with respect to $|G|$. \square

Now that we have a description of the alternating paths, we can use it to compute the boundary of the Morse complex.

Definition 4.5.5. Given two different faces $F, G \in \mathcal{L}_1(\mathcal{A})$, let

$$[F \rightarrow G] = \frac{[\langle D(F), p(F) \rangle : \langle C(G), G \rangle]}{[\langle D(F), p(F) \rangle : \langle C(F), F \rangle]},$$

where the incidence numbers on the right are taken in the Salvetti complex $\text{Sal}(\mathcal{A})$, and $D(F)$ is defined as in the proof of Lemma 4.5.4.

Lemma 4.5.6. Given two different faces $F, G \in \mathcal{F}_1(\mathcal{A})$ such that $F \rightarrow G$, we have

$$[F \rightarrow G] = \pm \prod t_\ell,$$

where the product is on the set of lines $\ell \neq |G|$ passing through $p(F)$, such that G and C_0 lie in opposite half-planes, whereas F and C_0 lie in the same closed half-plane (with respect to ℓ). The sign is $+1$ if $p(F) = p(G)$, and -1 otherwise.

Proof. Denote by $E(F)$ and $E(G)$ the chambers $C(F)^F$ and $C(G)^G$, respectively. Notice that $E(F) = D(F)^{p(F)}$, and therefore $[\langle D(F), p(F) \rangle : \langle C(F), F \rangle] = 1$. See Figure 4.2 for an example.

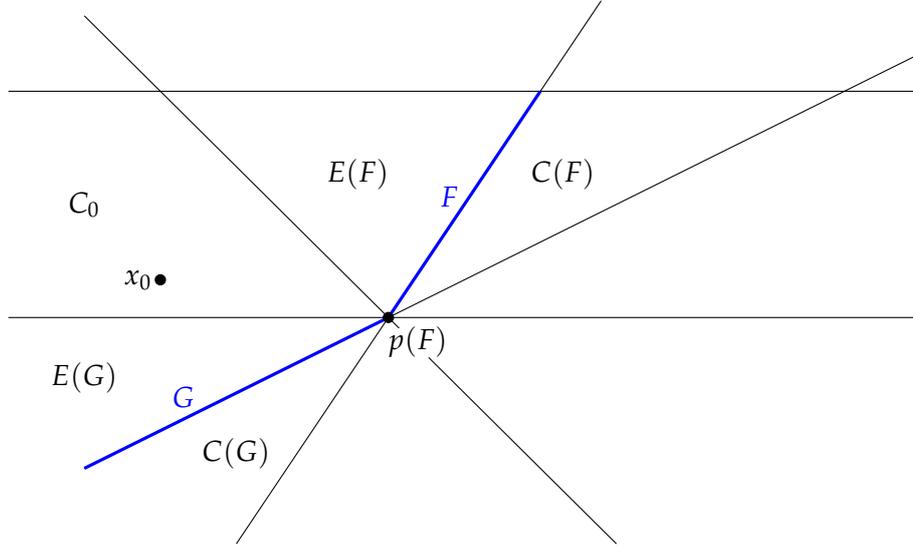


Figure 4.2: Faces $F, G \in \mathcal{F}_1(\mathcal{A})$ such that $F \rightarrow G$, as in Lemma 4.5.6.

Now we need to determine $\bar{u}(E(F), E(G))$, which is the product of the positive loops around the hyperplanes in $s(C_0, E(G)) \cap s(E(F), E(G))$. By definition of $E(G)$, we have that $s(C_0, E(G))$ is the set of lines different from $|G|$ for which G and C_0 in opposite half-planes. Since every line in $s(E(F), E(G))$ goes through $p(F)$, it is now easy to see that $s(C_0, E(G)) \cap s(E(F), E(G))$ is the set described in the statement.

We now need to determine the sign. If $p(G) = p(F)$, then we immediately see that G is in the half-plane delimited by $|F|$ that contains $D(F)$. The opposite is true if $p(G) \neq p(F)$. By our choice of the orientation, we obtain the stated result. \square

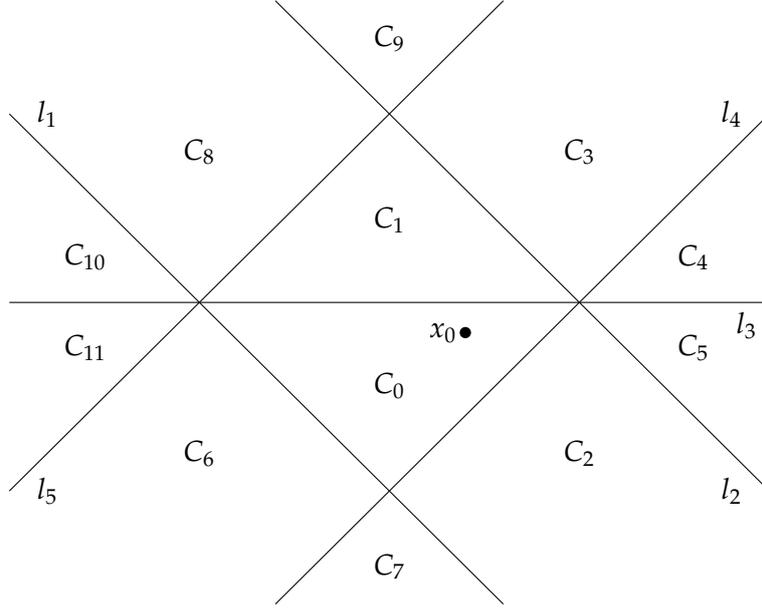
Theorem 4.5.7. Let \mathcal{A} be a locally finite line arrangement in \mathbb{R}^2 . Let $\langle D, p \rangle$ be a critical 2-cell and $\langle C, F \rangle$ a critical 1-cell. Then their incidence number in the Morse complex is given by

$$[\langle D, p \rangle : \langle C, F \rangle]^{\mathcal{M}} = \sum_{s \in \text{Seq}} \omega(s),$$

where Seq is the set of sequences of Lemma 4.5.4, and for each sequence $s = (F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_n = F) \in \text{Seq}$ we define

$$\omega(s) = (-1)^n [\langle D, p \rangle : \langle C(F_1), F_1 \rangle] \prod_{i=1}^{n-1} [F_i \rightarrow F_{i+1}].$$

Proof. It follows directly from the definition of the algebraic Morse complex, and from Lemma 4.5.4. \square


 Figure 4.3: Deconing A_3 .

	$\langle C_4, p_4 \rangle$	$\langle C_5, p_5 \rangle$	$\langle C_7, p_7 \rangle$	$\langle C_9, p_9 \rangle$	$\langle C_{10}, p_{10} \rangle$	$\langle C_{11}, p_{11} \rangle$
$\langle C_1, F_1 \rangle$	$1 - t_4$	$t_4(t_2 - 1)$	0	0	$t_1 - 1$	$t_1(1 - t_5)$
$\langle C_2, F_2 \rangle$	$t_2 t_3 - 1$	$t_2 - 1$	$1 - t_1$	0	0	0
$\langle C_3, F_3 \rangle$	$t_3(1 - t_4)$	$1 - t_3 t_4$	0	$t_5 - 1$	0	0
$\langle C_6, F_6 \rangle$	0	0	$t_4 - 1$	0	$1 - t_3 t_5$	$1 - t_5$
$\langle C_8, F_8 \rangle$	0	0	0	$1 - t_2$	$t_3(t_1 - 1)$	$t_1 t_3 - 1$

 Table 4.1: The boundary ∂_2 of the deconing of A_3

Example 4.5.8 (Deconing A_3). Consider the line arrangement \mathcal{A} of Figure 4.3, obtained by deconing the reflection arrangement of type A_3 . Given a chamber C_i , denote by $\langle C_i, F_i \rangle$ the associated critical cell if it has dimension 1, and by $\langle C_i, p_i \rangle$ if it has dimension 2. Applying Theorem 4.5.7 and Lemma 4.5.6, we obtain the boundary matrix ∂_2 of Table 4.1. This matrix is slightly simpler than the one computed in [GS09, Section 7], but there are many similarities. Specializing to the case $t_1 = \dots = t_5 = t$, we obtain that

$$H_1(M(\mathcal{A}); \mathbb{Q}[t^{\pm 1}]) \cong \left(\frac{\mathbb{Q}[t^{\pm 1}]}{t-1} \right)^3 \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{t^3-1},$$

as already computed for example in [GS09].

PART III

LOCAL HOMOLOGY

Precise matchings on weighted sheaves

5.1 INTRODUCTION

The next two chapters are devoted to the computation of the homology of Artin groups with local coefficients. The material of this chapter is based on a joint work with Mario Salvetti [PS18], and Chapter 6 is also based on [Pao19].

Assuming that the $K(\pi, 1)$ conjecture holds, the Salvetti complex [Sal87, Sal94] can be used to derive a combinatorial free resolution for Artin groups [DCS96] (equivalent resolutions also appeared in [Squ94] and [Ozo17, Pao17a]). Several (co)homology computations were carried out using this resolution [DCPSS99, DCPS01, CS04, Cal05, Cal06, CMS08a, CMS08b, CMS10]. The computational methods used in the previous papers are essentially based on filtering the algebraic complex and using the associated spectral sequence.

In [SV13] a more combinatorial method of calculation was introduced, based on the application of discrete Morse theory to a particular class of sheaves over posets (called *weighted sheaves*). Similar ideas were considered later in [CGN16], even if there the authors were mainly interested in computational aspects.

In the first part of this chapter (Sections 5.2, 5.3, and 5.4) we follow [SV13] and recall discrete Morse theory for weighted sheaves. Then, in Section 5.5, we expand this theory by introducing *precise matchings*. The motivation comes from the study of the local homology of spherical and affine Artin groups, as we will show in Chapter 6. The homology of the Morse complex associated to a precise matching is simpler to compute than in general (Theorems 5.5.2 and 5.5.4). In addition, the existence of precise matchings can be interpreted in terms of second-page collapsing of a spectral sequence which is naturally associated to the weighted sheaf (Proposition 5.5.3).

5.2 WEIGHTED SHEAVES OVER POSETS

Let (P, \preceq) be a finite poset.

Definition 5.2.1. Define a *sheaf of rings* over P (or a *diagram of rings* over P) as a collection

$$\{A_x \mid x \in P\}$$

of commutative rings, together with a collection of ring homomorphisms

$$\{\rho_{x,y}: A_y \rightarrow A_x \mid x \preceq y\}$$

satisfying $\rho_{x,x} = \text{id}_{A_x}$ and

$$x \preceq y \preceq z \Rightarrow \rho_{x,z} = \rho_{x,y} \circ \rho_{y,z}.$$

In other words, a sheaf of rings over P is a functor from P^{op} to the category of commutative rings.

Fix a PID R . We usually take $R = \mathbb{Q}[q^{\pm 1}]$. The divisibility relation gives R the structure of a small category, and R/R_* the structure of a poset (where R_* is the group of units in R) with bottom element the class of the units and top element 0. Any functor $w: (P, \preceq) \rightarrow (R, |)$, which maps every $x \in P$ to some $w(x) \in R \setminus \{0\}$, defines a sheaf over P by considering the collection of rings

$$\{R/(w(x)) \mid x \in P\},$$

with maps

$$\{i_{x,y}: R/(w(y)) \rightarrow R/(w(x)) \mid x \preceq y\},$$

where $i_{x,y}$ is induced by the identity of R .

Definition 5.2.2 (Weighted sheaf). Given a poset P , a PID R , and a morphism $w: (P, \preceq) \rightarrow (R, |)$ as above, we call the triple (P, R, w) a *weighted sheaf* over P and the coefficients $w(x)$ the *weights* of the sheaf.

Remark 5.2.3. Consider the *poset topology* over P , where a basis for the open sets is given by the upper ideals $\mathcal{B} = \{P_{>p}, p \in P\}$. Let (P, R, w) be a weighted sheaf. Then one can see w as a functor from (\mathcal{B}, \supseteq) to $(R, |)$ and thus a weighted sheaf defines a sheaf in the usual sense.

From now on our poset will be a simplicial complex K defined over a finite set S , with the partial order given by inclusion: $\sigma \preceq \tau$ if and only if $\sigma \subseteq \tau$. We adopt the convention that K contains the empty simplex \emptyset . A weighted sheaf over K is given by assigning to each simplex $\sigma \in K$ a weight $w(\sigma) \in R$, in such a way that $w(\sigma) \mid w(\tau)$ whenever $\sigma \preceq \tau$. Let $C_*^0(K; R)$ be the 1-shifted standard algebraic complex computing the simplicial homology of K :

$$C_k^0(K; R) = \bigoplus_{\substack{\sigma \in K \\ |\sigma|=k}} R e_\sigma^0,$$

where e_σ^0 is the generator associated to a given orientation of σ . In particular, $C_0^0(K; R) = R e_\emptyset^0$. The boundary is given by

$$\partial^0(e_\sigma^0) = \sum_{|\tau|=k-1} [\sigma : \tau] e_\tau^0,$$

where $[\sigma : \tau]$ is the incidence number (which is equal to ± 1 if $\tau \prec \sigma$, and vanishes otherwise).

Definition 5.2.4. The *weighted complex* associated to the weighted sheaf (K, R, w) is the algebraic complex $L_* = L_*(K, w)$ defined by

$$L_k = \bigoplus_{|\sigma|=k} \frac{R}{(w(\sigma))} \bar{e}_\sigma,$$

with boundary $\partial: L_k \rightarrow L_{k-1}$ induced by ∂^0 :

$$\partial(a_\sigma \bar{e}_\sigma) = \sum_{\tau \prec \sigma} [\sigma : \tau] i_{\tau, \sigma}(a_\sigma) \bar{e}_\tau.$$

There is a natural projection $\pi: C_*^0(K; R) \rightarrow L_*$ which maps a generator e_σ in C_*^0 to the generator \bar{e}_σ in L_* .

Remark 5.2.5. The diagram $\{R/(w(\sigma)) \mid \sigma \in K\}$ also defines a sheaf over the poset K in the sense of Remark 5.2.3. The sheaf cohomology associated to the open covering given by the upper ideals coincides with the homology of the weighted sheaf.

5.3 DECOMPOSITION AND FILTRATION

Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf. For any irreducible $\varphi \in R$, we define the φ -primary component $\mathcal{S}_\varphi = (K, R, w_\varphi)$ of the weighted sheaf \mathcal{S} by setting

$$w_\varphi(\sigma) = \varphi^{v_\varphi(\sigma)},$$

where $v_\varphi(\sigma)$ is the maximal r such that φ^r divides $w(\sigma)$. Since R is a PID and $w(\sigma) \neq 0$, such a maximal value exists. Notice that \mathcal{S}_φ is also a weighted sheaf. The weighted complex $(L_*)_\varphi$ associated to \mathcal{S}_φ is called the φ -primary component of the weighted complex L_* . In degree k we have:

$$(L_k)_\varphi = \bigoplus_{|\sigma|=k} \frac{R}{(\varphi^{v_\varphi(\sigma)})} \bar{e}_\sigma.$$

The complex $(L_*)_\varphi$ has a natural increasing filtration by the subcomplexes

$$F^s(L_*)_\varphi = \bigoplus_{v_\varphi(\sigma) \leq s} \frac{R}{(\varphi^{v_\varphi(\sigma)})} \bar{e}_\sigma.$$

This filtration is associated to an increasing filtration of the simplicial complex K :

$$K_{\varphi,s} = \{\sigma \in K \mid v_\varphi(\sigma) \leq s\}.$$

Then $F^s(L_*)_\varphi$ is the weighted complex associated to the weighted sheaf

$$(K_{\varphi,s}, R, w_\varphi|_{K_{\varphi,s}}).$$

Theorem 5.3.1 ([SV13]). Let (K, R, w) be a weighted sheaf, with associated weighted complex L_* . For any irreducible $\varphi \in R$, there exists a spectral sequence

$$E_{p,q}^r \Rightarrow H_*((L_*)_\varphi)$$

that abuts to the homology of the φ -primary component of the associated algebraic complex L_* . Moreover the E^1 -term

$$E_{p,q}^1 = H_{p+q}(F^p/F^{p-1}) \cong H_{p+q}(K_{\varphi,p}, K_{\varphi,p-1}; R/(\varphi^p))$$

is isomorphic to the relative homology with trivial coefficients of the simplicial pair $(K_{\varphi,p}, K_{\varphi,p-1})$.

5.4 DISCRETE MORSE THEORY FOR WEIGHTED COMPLEXES

We are now going to describe a variant of discrete Morse theory (see Sections 1.2 and 1.3) which applies to weighted sheaves.

Definition 5.4.1. A *weighted acyclic matching* on a weighted sheaf (P, R, w) over P is an acyclic matching \mathcal{M} on P such that

$$(x, y) \in \mathcal{M} \Rightarrow (w(x)) = (w(y)).$$

Standard discrete Morse theory generalizes as follows. Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over a finite simplicial complex K , and let \mathcal{M} an acyclic weighted matching. A *critical simplex* of K is a simplex σ which does not belong to any pair of \mathcal{M} . The following definition is equivalent to [SV13, Definition 3.3].

Definition 5.4.2. The *Morse complex* of \mathcal{S} with respect to \mathcal{M} is defined as the torsion complex

$$L_*^{\mathcal{M}} = \bigoplus_{\sigma \text{ critical}} \frac{R}{(w(\sigma))} \bar{e}_\sigma$$

with boundary

$$\partial^{\mathcal{M}}(\bar{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau| = |\sigma| - 1}} [\sigma : \tau]^{\mathcal{M}} \bar{e}_\tau,$$

where $[\sigma : \tau]^{\mathcal{M}} \in \mathbb{Z}$ is given by the sum over all alternating paths

$$\sigma = \sigma_0 \succ \tau_1 \prec \sigma_1 \succ \cdots \succ \tau_k \prec \sigma_k \succ \tau,$$

from σ to τ of the quantity

$$(-1)^k [\sigma_k : \tau] \prod_{i=1}^k [\sigma_{i-1} : \tau_i] [\sigma_i : \tau_i]$$

(see Theorem 1.2.5). The boundary map $\partial^{\mathcal{M}}$ is extended by R -linearity, where some care should be taken since each \bar{e}_τ lives in a component with a possibly different torsion.

Theorem 5.4.3 ([SV13]). Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over a simplicial complex K . Let \mathcal{M} be an acyclic matching on \mathcal{S} . Then there is an isomorphism

$$H_*(L_*, \partial) \cong H_*(L_*^{\mathcal{M}}, \partial^{\mathcal{M}})$$

between the homology of the weighted complex L_* associated to \mathcal{S} and the homology of the Morse complex $L_*^{\mathcal{M}}$ of \mathcal{S} .

Remark 5.4.4. If we forget about the weights, the matching \mathcal{M} is in particular an acyclic matching on the simplicial complex K . Therefore the algebraic complex $C_*^0(K; A)$, which computes the (shifted and reduced) simplicial homology of K with coefficients in some ring A , admits a “classical” algebraic Morse complex

$$C_*^0(K; A)^{\mathcal{M}} = \bigoplus_{\sigma \text{ critical}} A \bar{e}_\sigma$$

with boundary map

$$\delta^{\mathcal{M}}(\bar{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau| = |\sigma| - 1}} [\sigma : \tau]^{\mathcal{M}} \bar{e}_\tau.$$

Remark 5.4.5. Set $C_*^0 = C_*^0(K; R)$, and let $\pi: C_*^0 \rightarrow L_*$ be the natural projection. The composition with the projection $L_* \rightarrow (L_*)_\varphi$, for some fixed irreducible element $\varphi \in R$, yields a natural projection $\pi_\varphi: C_*^0 \rightarrow (L_*)_\varphi$. Similarly, at the level of the Morse complexes, we have a projection $\bar{\pi}_\varphi: (C_*^0)^{\mathcal{M}} \rightarrow (L_*)_\varphi^{\mathcal{M}}$ which sends a generator \bar{e}_σ in $(C_*^0)^{\mathcal{M}}$ to the generator \bar{e}_σ in $(L_*)_\varphi^{\mathcal{M}}$. Then, by construction, the maps induced in homology by the projections π_φ and $\bar{\pi}_\varphi$ commute with the Morse isomorphisms of [Koz07] and [SV13]:

$$\begin{array}{ccc} H_*(C_*^0) & \xrightarrow{(\pi_\varphi)_*} & H_*((L_*)_\varphi) \\ \cong \downarrow & & \cong \downarrow \\ H_*((C_*^0)^{\mathcal{M}}) & \xrightarrow{(\bar{\pi}_\varphi)_*} & H_*((L_*)_\varphi^{\mathcal{M}}). \end{array}$$

5. PRECISE MATCHINGS ON WEIGHTED SHEAVES

Let $\varphi \in R$ be an irreducible element, and let \mathcal{M} be a weighted acyclic matching on \mathcal{S}_φ . The filtration on the weighted complex induces a filtration on the Morse complex:

$$F^p(L_*)_{\varphi}^{\mathcal{M}} = \bigoplus_{\substack{\sigma \text{ critical} \\ v_{\varphi}(\sigma) \leq p}} \frac{R}{(\varphi^{v_{\varphi}(\sigma)})} \bar{e}_{\sigma}.$$

Consider also the quotient complex

$$\mathcal{F}^p(L_*)_{\varphi}^{\mathcal{M}} = F^p(L_*)_{\varphi}^{\mathcal{M}} / F^{p-1}(L_*)_{\varphi}^{\mathcal{M}} \cong \bigoplus_{\substack{\sigma \text{ critical} \\ v_{\varphi}(\sigma) = p}} \frac{R}{(\varphi^p)} \bar{e}_{\sigma}.$$

Theorem 5.4.6 ([SV13]). Let \mathcal{S} and φ be as above, and let \mathcal{M} be an acyclic matching on \mathcal{S}_φ . Then the E^1 -page of the spectral sequence of Theorem 5.3.1 is identified with

$$E_{p,q}^1 \cong H_{p+q}(\mathcal{F}^p(L_*)_{\varphi}^{\mathcal{M}}),$$

where $(L_*)_{\varphi}^{\mathcal{M}}$ is the Morse complex of \mathcal{S}_φ . The differential

$$d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$$

is induced by the boundary of the Morse complex, and thus it can be also computed by using alternating paths.

5.5 PRECISE MATCHINGS

In this section we are going to introduce *precise matchings* on weighted sheaves. The motivation comes from the study of the local homology of spherical and affine Artin groups, as we will show in Chapter 6.

Assume from now on that the PID R contains some field \mathbb{K} . Our main case of interest is $\mathbb{K} = \mathbb{Q}$ and $R = \mathbb{Q}[q^{\pm 1}]$. Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over the finite simplicial complex K , with associated weighted complex L_* . Given a fixed irreducible element φ of R , let \mathcal{S}_φ be the φ -primary component of \mathcal{S} and let $(L_*)_{\varphi}$ be its associated weighted complex. Let \mathcal{M} be a weighted acyclic matching on \mathcal{S}_φ .

Let $G^{\mathcal{M}}$ be the Hasse diagram of the corresponding Morse complex: the vertices of $G^{\mathcal{M}}$ are the critical simplices of K , and there is an (oriented) edge $\sigma \rightarrow \tau$ whenever $[\sigma : \tau]^{\mathcal{M}}$ is not 0 in R (or equivalently in \mathbb{K}), where $[\sigma : \tau]^{\mathcal{M}} \in \mathbb{Z}$ is the incidence number between σ and τ in the Morse complex of K . In other words, there is an edge $\sigma \rightarrow \tau$ if $[\sigma : \tau]^{\mathcal{M}}$ is not a multiple of $\text{char } \mathbb{K} = \text{char } R$. When $\mathbb{K} = \mathbb{Q}$, this simply means that $[\sigma : \tau]^{\mathcal{M}} \neq 0$.

Let \mathcal{I} be the set of connected components of $G^{\mathcal{M}}$ (computed ignoring the orientation of the edges). Recall that $v_{\varphi}(\sigma)$ is the maximal $k \in \mathbb{N}$ such that φ^k divides $w(\sigma)$.

Definition 5.5.1. The matching \mathcal{M} is φ -precise (or simply *precise*) if, for any edge $\sigma \rightarrow \tau$ of $G^{\mathcal{M}}$, we have that $v_{\varphi}(\sigma) = v_{\varphi}(\tau) + 1$.

In other words \mathcal{M} is precise if, for any two simplices σ and τ lying in the same connected component $i \in \mathcal{I}$, the following relation holds:

$$v_{\varphi}(\sigma) - v_{\varphi}(\tau) = |\sigma| - |\tau|.$$

Equivalently the quantity $|\sigma| - v_{\varphi}(\sigma)$, as a function of σ , is constant within a fixed connected component of $G^{\mathcal{M}}$. This definition is motivated by the fact that precise matchings exist in many cases of interest, as we will see in Chapter 6, and that the homology of the Morse complex is much simpler to compute (and takes a particularly nice form) when the matching is precise. The name “precise” has been chosen because for a generic matching one only has $v_{\varphi}(\sigma) \geq v_{\varphi}(\tau)$ (when $\sigma \rightarrow \tau$ is an edge of $G^{\mathcal{M}}$), and we require $v_{\varphi}(\sigma)$ to be precisely $v_{\varphi}(\tau) + 1$.

Assume from now on that \mathcal{M} is a φ -precise matching. To simplify the notation, set $(A_*, \partial) = ((L_*)_{\varphi}^{\mathcal{M}}, \partial^{\mathcal{M}})$ and $(V_*, \delta) = (C_*^0(K, \mathbb{K})^{\mathcal{M}}, \delta^{\mathcal{M}})$. Our aim is to derive a formula for the homology of the Morse complex A_* . Since the differential δ vanishes between simplices in different connected components of $G^{\mathcal{M}}$, the complex (V_*, δ) splits as

$$(V_*, \delta) = \bigoplus_{i \in \mathcal{I}} (V_*^i, \delta^i),$$

where

$$V_*^i = \bigoplus_{\substack{\sigma \text{ critical} \\ \sigma \in i}} \mathbb{K} \bar{e}_{\sigma}$$

and the boundary map $\delta^i: V_*^i \rightarrow V_*^i$ is the restriction of δ to V_*^i . The differential ∂ of A_* is induced by δ , and thus it also vanishes between simplices in different connected components. Therefore we have an analogous splitting for (A_*, ∂) :

$$(A_*, \partial) = \bigoplus_{i \in \mathcal{I}} (A_*^i, \partial^i),$$

where

$$A_*^i = \bigoplus_{\substack{\sigma \text{ critical} \\ \sigma \in i}} \frac{R}{(w_{\varphi}(\sigma))} \bar{e}_{\sigma}$$

and the boundary map $\partial^i: A_*^i \rightarrow A_*^i$ is simply the restriction of ∂ to A_*^i .

Fix now a connected component $i \in \mathcal{I}$. Since \mathcal{M} is precise, there exists some $k \in \mathbb{Z}$ (which depends on i) such that $v_{\varphi}(\sigma) = |\sigma| + k$ for all $\sigma \in i$. Therefore in degree m we have

$$A_m^i = \bigoplus_{\sigma \in \text{Cr}_m^i} \frac{R}{(\varphi^{m+k})} \bar{e}_{\sigma} = \left(\bigoplus_{\sigma \in \text{Cr}_m^i} \mathbb{K} \bar{e}_{\sigma} \right) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} = V_m^i \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})},$$

5. PRECISE MATCHINGS ON WEIGHTED SHEAVES

where Cr_m^i is the set of critical simplices σ such that $\sigma \in i$ and $|\sigma| = m$. By construction the boundary $\partial_m^i: A_m^i \rightarrow A_{m-1}^i$ factors accordingly:

$$\partial_m^i = \delta_m^i \otimes_{\mathbb{K}} \pi_m,$$

where

$$\pi_m: \frac{R}{(\varphi^{m+k})} \rightarrow \frac{R}{(\varphi^{m+k-1})}$$

is the projection induced by the identity $R \rightarrow R$. Since $\text{im } \delta_{m+1}^i \subseteq \ker \delta_m^i$, each V_m^i splits (as a vector space over \mathbb{K}) as a direct sum of linear subspaces:

$$V_m^i = W_{m,1}^i \oplus W_{m,2}^i \oplus W_{m,3}^i,$$

where $W_{m,1}^i = \text{im } \delta_{m+1}^i$ and $W_{m,1}^i \oplus W_{m,2}^i = \ker \delta_m^i$. Then

$$\begin{aligned} \ker(\partial_m^i) &= \ker(\delta_m^i \otimes_{\mathbb{K}} \pi_m) \\ &= \left((W_{m,1}^i \oplus W_{m,2}^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(W_{m,3}^i \otimes_{\mathbb{K}} \frac{(\varphi^{m+k-1})}{(\varphi^{m+k})} \right) \\ \text{im}(\partial_{m+1}^i) &= \text{im}(\delta_{m+1}^i \otimes_{\mathbb{K}} \pi_{m+1}) \\ &= W_{m,1}^i \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})}. \end{aligned}$$

Therefore the homology of (A_*^i, δ^i) is given, as an R -module, by

$$\begin{aligned} H_m(A_*^i) &= \frac{\ker(\partial_m^i)}{\text{im}(\partial_{m+1}^i)} \\ &= \left(W_{m,2}^i \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(W_{m,3}^i \otimes_{\mathbb{K}} \frac{(\varphi^{m+k-1})}{(\varphi^{m+k})} \right) \\ &\cong \left(H_m(V_*^i, \delta^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(\mathbb{K}^{\text{rk } \delta_m^i} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \end{aligned}$$

In the last isomorphism we used the fact that $\dim W_{m,3}^i = \dim V_m^i - \dim \ker \delta_m^i = \text{rk } \delta_m^i$.

Recall that the previous formula holds for a fixed connected component $i \in \mathcal{I}$, and k depends on i . Since we now need to take the direct sum over the connected components, let k_i be the value of k for the component i .

Theorem 5.5.2. The homology of $(L_*)_{\varphi}$ is given, as an R -module, by

$$H_m((L_*)_{\varphi}, \partial) \cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V_*^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\mathbb{K}^{\text{rk } \delta_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right).$$

Proof. By Theorem 5.4.3, $H_m((L_*)_{\varphi}, \partial) \cong H_m((L_*)_{\varphi}^{\mathcal{M}}, \partial^{\mathcal{M}})$. Using what we have done in this section, we have that

$$\begin{aligned} H_m((L_*)_{\varphi}^{\mathcal{M}}, \partial^{\mathcal{M}}) &= H_m(A_*, \partial) = \bigoplus_{i \in \mathcal{I}} H_m(A_*^i, \partial) \\ &\cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V_*^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\left(\bigoplus_{i \in \mathcal{I}} \mathbb{K}^{\text{rk } \delta_m^i} \right) \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right) \\ &\cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V_*^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\mathbb{K}^{\text{rk } \delta_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \quad \square \end{aligned}$$

We now show how the existence of a precise matching can be interpreted in terms of the spectral sequence associated to the weighted sheaf (see Theorem 5.3.1).

Proposition 5.5.3. If a φ -precise matching \mathcal{M} exists, then the spectral sequence $E_{p,q}^r$ associated to the weighted sheaf \mathcal{S}_{φ} collapses at the E^2 -page.

Proof. By Theorem 5.4.6, the E^1 -page can be computed through the Morse complex of our matching \mathcal{M} :

$$E_{p,q}^1 \cong H_{p+q}(\mathcal{F}^p(L_*)_{\varphi}^{\mathcal{M}}),$$

and the differential $d_{p,q}^1$ is induced by the boundary of the Morse complex. The spectral sequence then splits as a direct sum over the connected components of $G^{\mathcal{M}}$:

$$E_{p,q}^r = \bigoplus_{i \in \mathcal{I}} E_{p,q}^{r,i}$$

where $E_{p,q}^{1,i} \cong H_{p+q}(\mathcal{F}^p A_*^i)$. Since the matching is precise, for $m \neq p - k_i$ we have

$$\mathcal{F}^p A_m^i = F^p A_m^i / F^{p-1} A_m^i = 0.$$

This means that the page $E_{p,q}^{0,i} \cong E_{p,q}^{1,i}$ is non-trivial only in the row $q = -k_i$, and the entire spectral sequence $E_{p,q}^r$ collapses at the E^2 -page. \square

What we have done so far in this section assumed φ to be some fixed irreducible element of the PID R . In order to recover the full homology of L_* , we need to let φ vary among all equivalence classes of irreducible elements of R modulo the units. Suppose from now on that we have a φ -precise matching \mathcal{M}_{φ} on \mathcal{S}_{φ} , for each φ . The following result follows immediately from Theorem 5.5.2, provided that we add a “ φ ” subscript (or superscript) to all the quantities that depend on the matching \mathcal{M}_{φ} .

Theorem 5.5.4. The homology of L_* is given, as an R -module, by

$$H_m(L_*, \partial) \cong \bigoplus_{\varphi} \left(\bigoplus_{i \in \mathcal{I}_{\varphi}} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right) \oplus \left(\mathbb{K}^{\text{rk } \delta_m^{\varphi}} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \quad \square$$

For later applications, we finally need to study how the isomorphism of Theorem 5.5.4 behaves with respect to the projection $\pi: C_*^0 \rightarrow L_*$ of Remark 5.4.5. This projection is the direct sum over φ of the projections

$$\pi_\varphi: C_*^0 \rightarrow (L_*)_\varphi.$$

Instead of studying the induced map $(\pi_\varphi)_*: H_m(C_*^0) \rightarrow H_m((L_*)_\varphi)$, we study the map

$$(\bar{\pi}_\varphi)_*: H_m((C_*^0)^\mathcal{M}) \rightarrow H_m((L_*)_\varphi^\mathcal{M})$$

between the Morse complexes — here $\mathcal{M} = \mathcal{M}_\varphi$ is a precise matching which depends on φ . For $i \in \mathcal{I}_\varphi$, let $\pi_i: (C_*^0)^\mathcal{M} \rightarrow V_*^{\varphi,i} \otimes_{\mathbb{K}} R \subseteq (C_*^0)^\mathcal{M}$ be the projection onto the subcomplex corresponding to the connected component i , and let $(\pi_i)_*$ be the map induced in homology. Let $[c] \in H_m((C_*^0)^\mathcal{M})$, for some cycle $c \in \ker \delta^\mathcal{M} \subseteq (C_m^0)^\mathcal{M}$. Applying the map $(\bar{\pi}_\varphi)_*: H_m((C_*^0)^\mathcal{M}) \rightarrow H_m((L_*)_\varphi^\mathcal{M})$, we obtain

$$\begin{aligned} (\bar{\pi}_\varphi)_*([c]) &= (\bar{\pi}_\varphi)_* \left(\sum_{i \in \mathcal{I}_\varphi} (\pi_i)_*([c]) \right) \\ &= \sum_{i \in \mathcal{I}_\varphi} (\bar{\pi}_\varphi)_* \left((\pi_i)_*([c]) \right). \end{aligned}$$

Applying the isomorphism of Theorem 5.5.2, this element is sent to

$$\sum_{i \in \mathcal{I}_\varphi} \left((\pi_i)_*([c]) \right) \otimes_{\mathbb{K}} [1] \in \bigoplus_{i \in \mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}.$$

We are going to use these formulas to prove the following result, which describes the kernel and the cokernel of π_* .

Proposition 5.5.5. The cokernel of $\pi_*: H_m(C_*^0) \rightarrow H_m(L_*)$ is given by

$$\text{coker } \pi_* \cong \bigoplus_{\varphi} \left(\frac{R}{(\varphi)} \right)^{\oplus \text{rk } \delta_m^\varphi}.$$

In addition, the kernel of π_* is a free R -module isomorphic to $H_m(C_*^0)$.

Proof. Throughout the proof, consider the following R -modules identified one with each other, without explicitly mentioning the isomorphisms between them:

$$H_m(L_*) \cong \bigoplus_{\varphi} H_m((L_*)_\varphi)$$

$$\begin{aligned} &\cong \bigoplus_{\varphi} H_m((L_*)_{\varphi}^{\mathcal{M}}) \\ &\cong \bigoplus_{\varphi} \left(\bigoplus_{i \in \mathcal{I}_{\varphi}} H_m(V_*^{\varphi, i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi, i}})} \right) \oplus \left(\mathbb{K}^{\text{rk } \delta_m^{\varphi}} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \end{aligned}$$

Recall that the matching \mathcal{M} depends on φ , although we write \mathcal{M} instead of \mathcal{M}_{φ} in order to make the notation more readable. Also recall that the isomorphisms $H_m((L_*)_{\varphi}) \cong H_m((L_*)_{\varphi}^{\mathcal{M}})$ occur in the commutative diagram of Remark 5.4.5.

We want to show that the image of $\pi_*: H_m(C_*^0) \rightarrow H_m(L_*)$ is given by

$$\text{im } \pi_* = \bigoplus_{\varphi} \left(\bigoplus_{i \in \mathcal{I}_{\varphi}} H_m(V_*^{\varphi, i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi, i}})} \right) \subseteq H_m(L_*).$$

Let $\psi_{\varphi}: H_*(C_*^0) \rightarrow H_*((L_*)_{\varphi}^{\mathcal{M}})$ be the map defined as the composition

$$H_*(C_*^0) \xrightarrow{\cong} H_*((C_*^0)^{\mathcal{M}}) \xrightarrow{(\bar{\pi}_{\varphi})_*} H_*((L_*)_{\varphi}^{\mathcal{M}}).$$

By commutativity of the diagram of Remark 5.4.5, the image of $\pi_* = \bigoplus_{\varphi} (\pi_{\varphi})_*$ is the same as the image of

$$\bigoplus_{\varphi} \psi_{\varphi}: H_*(C_*^0) \rightarrow \bigoplus_{\varphi} H_*((L_*)_{\varphi}^{\mathcal{M}}).$$

We have already proved that, for any $[c] \in H_m((C_*^0)^{\mathcal{M}})$,

$$(\bar{\pi}_{\varphi})_*([c]) = \sum_{i \in \mathcal{I}_{\varphi}} \left((\pi_i)_*([c]) \right) \otimes_R [1] \in \bigoplus_{i \in \mathcal{I}_{\varphi}} H_m(V_*^{\varphi, i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi, i}})},$$

which means in particular that

$$\text{im } (\bar{\pi}_{\varphi})_* \subseteq \bigoplus_{i \in \mathcal{I}_{\varphi}} H_m(V_*^{\varphi, i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi, i}})}.$$

Therefore we immediately have the inclusion

$$\text{im } (\pi_*) \subseteq \sum_{\varphi} \text{im } \psi_{\varphi} = \sum_{\varphi} \text{im } (\bar{\pi}_{\varphi})_* \subseteq \bigoplus_{\varphi} \left(\bigoplus_{i \in \mathcal{I}_{\varphi}} H_m(V_*^{\varphi, i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi, i}})} \right).$$

To prove the opposite inclusion, we show that any element of the form

$$[c] \otimes_{\mathbb{K}} [1] \in H_m(V_*^{\varphi, i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi, i}})}$$

is in the image of π_* (for any fixed φ and i). To do so, choose $\alpha \in R$ such that $\alpha \equiv 1 \pmod{\varphi^{m+k_{\varphi,i}}}$ and $\alpha \equiv 0 \pmod{\eta^{m+k_{\eta,j}}}$ for any irreducible element $\eta \neq \varphi$ which divides some weight $w(\sigma)$, and for any connected component $j \in \mathcal{I}_\eta$ — there is only a finite number of such η 's up to multiplication by units, because K is finite. The element $c \otimes_{\mathbb{K}} \alpha$ is a cycle in $V_m^{\varphi,i} \otimes_{\mathbb{K}} R \subseteq C_m^0(K, \mathbb{K})^{\mathcal{M}} \otimes_{\mathbb{K}} R \cong (C_m^0)^{\mathcal{M}}$. Then, if $[\tilde{c}]$ is the preimage of $[c \otimes_{\mathbb{K}} \alpha]$ under the isomorphism $H_*(C_*^0) \xrightarrow{\cong} H_*((C_*^0)^{\mathcal{M}})$, we have that:

$$\begin{aligned} \psi_\varphi([\tilde{c}]) &= (\tilde{\pi}_\varphi)_*([c \otimes_{\mathbb{K}} \alpha]) = [c] \otimes_{\mathbb{K}} [1] \in H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \\ \psi_\eta([\tilde{c}]) &= (\tilde{\pi}_\eta)_*([c' \otimes_{\mathbb{K}} \alpha]) = 0 \quad \text{for any } \eta \neq \varphi, \end{aligned}$$

where $[c']$ is the image of $[c]$ under the isomorphism

$$H_*((C_*^0)^{\mathcal{M}_\varphi}) \xrightarrow{\cong} H_*(C_*^0) \xrightarrow{\cong} H_*((C_*^0)^{\mathcal{M}_\eta}).$$

Therefore $[c] \otimes_{\mathbb{K}} [1]$ is in the image of π_* . We have thus proved that

$$\text{im } \pi_* = \bigoplus_{\varphi} \left(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right).$$

Then the cokernel of π_* can be easily computed:

$$\text{coker } \pi_* = \frac{H_m(L_*)}{\text{im } \pi_*} = \bigoplus_{\varphi} \left(\mathbb{K}^{\text{rk } \delta_m^\varphi} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right) = \bigoplus_{\varphi} \left(\frac{R}{(\varphi)} \right)^{\oplus \text{rk } \delta_m^\varphi}.$$

The R -module $H_m(C_*^0) \cong H_m(C_*^0(K; \mathbb{K})) \otimes_{\mathbb{K}} R$ is free and finitely generated, because $H_m(C_*^0(K; \mathbb{K}))$ is a finite-dimensional vector space over \mathbb{K} (recall that K is a finite simplicial complex). The kernel of π_* is a submodule of $H_m(C_*^0)$, so it is itself a free R -module with lower or equal rank. Let $[c_1], \dots, [c_k]$ be an R -base of $H_m(C_*^0)$. Consider the non-zero ideal

$$I = \bigcap_{\varphi} \bigcap_{i \in \mathcal{I}_\varphi} (\varphi^{m+k_i}) \subseteq R,$$

where φ varies among the (finitely many) irreducible elements which divide some weight $w(\sigma)$ (for $\sigma \in K$). Fix any non-zero element $\alpha \in I$. Then the elements $\alpha[c_1], \dots, \alpha[c_k]$ generate a free submodule of $\ker \pi_*$ of rank $k = \text{rk } H_m(C_*^0)$. Therefore $\ker \pi_*$ and $H_m(C_*^0)$ have the same rank, so they are isomorphic as R -modules. \square

Local homology of Artin groups

6.1 INTRODUCTION

In this chapter we use weighted sheaves to study the local homology of Artin groups. The material is based on [PS18, Pao19], the first paper being a joint work with Mario Salvetti.

In Section 6.2 we recall how to naturally associate a weighted sheaf to every Artin group, as explained in [SV13]. Then the homology of the sheaf is equal to (the torsion part of) the homology of the Artin group with coefficients in a certain local system.

In Section 6.3 we point out a surprising relationship between the local homology of braid groups (Artin groups of type A_n) and the homology of certain independence complexes, which in their simplest form have already been studied in a combinatorial context (see for example [BK07, Koz07, Eng09]). The exact formula we are going to prove is the following.

Theorem 6.3.4. We have

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong \tilde{H}_{*-d+1} \left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)} \right),$$

where on the left there is the homology with local coefficients of the classical braid group (and $R = \mathbb{Q}[t^{\pm 1}]$), and on the right there is the homology with trivial coefficients of a certain independence complex. We denote by φ_d the d -th cyclotomic polynomial.

In Section 6.4 we show that the existence of precise matchings for a certain Artin group has strong implications for its homology. In particular we prove the following result.

Theorem 6.4.2. Let G_W be an Artin group that admits a φ -precise matching for all cyclotomic polynomials $\varphi = \varphi_d$ (with $d \geq 2$). Then the local homology $H_*(X_W; R)$ does not have φ^k -torsion for $k \geq 2$.

Here X_W is the Salvetti complex, introduced in Section 2.6. Recall that it is a finite CW complex which is a deformation retract of the orbit space Y_W associated to W , and is conjectured to be a $K(G_W, 1)$. Theorem 6.4.2 can be seen as a result about the homology of an infinite cyclic covering of X_W and its monodromy. The conclusion of this theorem was already known for all Artin groups of spherical type, thanks to a geometric argument (see Remark 6.4.3). Our construction provides a new combinatorial condition that applies to a wider class of Artin groups.

In the rest of this chapter we show that precise matchings exist for all Artin groups of spherical and affine type. The main results, stated in Section 6.4, are the following.

Theorem 6.4.4. Every Artin group of spherical or affine type admits a φ -precise matching for each cyclotomic polynomial φ .

Corollary 6.4.5. Let G_W be an Artin group of spherical or affine type. Then the local homology $H_*(X_W; R)$ has no φ^k -torsion for $k \geq 2$.

We use precise matchings to carry out explicit homology computations for some of the infinite families $(A_n, B_n, \tilde{A}_n, \tilde{C}_n)$ and for all exceptional cases. In particular, the computation of the local homology of affine Artin groups of type \tilde{C}_n is new.

Theorem 6.12.5. Let G_W be an Artin group of type \tilde{C}_n . Then the φ_d -primary component of $H_*(X_W; R)$ is trivial for d odd, and for d even is as follows:

$$H_m(X_W; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus m+k-n+1} & \text{if } n-k \leq m \leq n-1 \\ 0 & \text{otherwise,} \end{cases}$$

where $n = k\frac{d}{2} + r$.

We recover the results of [DCPSS99, SV13] on the local homology of exceptional groups, with small corrections (see Section 6.14).

We also provide a software library which can be used to generate matchings for any spherical or affine Artin group, check preciseness, and compute the local homology [Pao17b].

6.2 WEIGHTED SHEAVES FOR ARTIN GROUPS

Let (W, S) be a Coxeter system, with associated Coxeter graph Γ and Artin group G_W . Recall from Section 2.6 the definition of the configuration spaces Y and $Y_W = Y/W$, and of the Salvetti complex X_W , whose cells are indexed by the simplicial complex

$$\mathcal{K}_W = \{\sigma \subseteq S \mid \text{the parabolic subgroup } W_\sigma \text{ is finite}\}.$$

6.2. Weighted sheaves for Artin groups

Consider the action of the Artin group G_W on the ring $R = \mathbb{Q}[q^{\pm 1}]$ given by

$$g_s \mapsto [\text{multiplication by } -q] \quad \text{for all } s \in S.$$

Then the homology $H_*(X_W; R)$ with local coefficients is computed by the algebraic complex

$$C_k = \bigoplus_{\substack{\sigma \in \mathcal{K}_W \\ |\sigma|=k}} R e_\sigma$$

with boundary

$$\partial(e_\sigma) = \sum_{\tau \triangleleft \sigma} [\sigma : \tau] \frac{W_\sigma(q)}{W_\tau(q)} e_\tau,$$

where $W_\sigma(q) = \sum_{w \in W_\sigma} q^{l(w)}$ is the Poincaré polynomial of W_σ (see [Sal94]).

Remark 6.2.1. The R -module structure on $H_*(X_W; R)$ is given by the transformation μ_q induced by q -multiplication. If the order of μ_q is N , then the homology groups decompose into cyclic factors which are either free, of the form R^k , or torsion, of the form $R/(\varphi_d)$, where φ_d is the d -th cyclotomic polynomial and $d \mid N$. Therefore we are interested in localizing to cyclotomic polynomials.

Theorem 6.2.2 ([SV13]). To a Coxeter system (W, S) we can associate a weighted sheaf (K, R, w) over the simplicial complex $K = \mathcal{K}_W$, by setting $w(\sigma) = W_\sigma(q)$. Also, for any cyclotomic polynomial $\varphi = \varphi_d$, the map $w_\varphi(\sigma)$ — which gives the maximal power of φ that divides $W_\sigma(q)$ — defines a weighted sheaf (K, R, w_φ) over $K = \mathcal{K}_W$.

The homology of the associated weighted complex is strictly related to the homology of X_W . Specifically, set $C_*^0 = C_*^0(K; R)$ and consider the diagonal map

$$\Delta: C_* \rightarrow C_*^0, \quad e_\sigma \mapsto W_\sigma(q) e_\sigma^0.$$

By the formula for the boundary map it follows that Δ is an injective chain-complex homomorphism, so there is an exact sequence of complexes:

$$0 \longrightarrow C_* \xrightarrow{\Delta} C_*^0 \xrightarrow{\pi} L_* \longrightarrow 0,$$

where

$$L_k = \bigoplus_{\substack{\sigma \in \mathcal{K}_W \\ |\sigma|=k}} \frac{R}{(W_\sigma(q))} \bar{e}_\sigma$$

is the quotient complex. Passing to the associated long exact sequence we get:

$$\cdots \xrightarrow{\pi_*} H_{k+1}(L_*) \rightarrow H_k(C_*) \xrightarrow{\Delta_*} H_k(C_*^0) \xrightarrow{\pi_*} H_k(L_*) \rightarrow \cdots \quad (6.1)$$

Then the homology of L_* can be used to compute the homology of C_* .

The orbit space Y_W (and thus the CW-complex $X_W \simeq Y_W$) is conjectured to be a classifying space for the Artin group G_W (Conjecture 2.6.1). Whenever the conjecture holds, the homology $H_*(X_W; R)$ coincides with the local homology $H_*(G_W; R)$ of the Artin group G_W .

6.3 BRAID GROUPS AND INDEPENDENCE COMPLEXES

In this section we are going to show how the local homology of braid groups (i.e. Artin groups of type A_n) is related to the homology of suitable independence complexes. Recall that the Coxeter graph of type A_n is a linear graph with n vertices, usually labeled $1, 2, \dots, n$ (see Figure 2.4), and that the corresponding Artin group is the braid group on $n + 1$ strands, which we denote by Br_{n+1} . With a slight abuse of notation, by A_n we will sometimes indicate the graph itself.

The homology of Br_{n+1} , with coefficients in the representation $R = \mathbb{Q}[q^{\pm 1}]$ described in the previous section, was computed in [Fre88, DCPS01]. See also [Cal06] for coefficients in $\mathbb{Z}[q^{\pm 1}]$ (but we will not address this case).

Theorem 6.3.1 ([Fre88, DCPS01]). The φ_d -primary component of the local homology of a braid group is given by

$$H_*(\text{Br}_{n+1}; R)_{\varphi_d} = \begin{cases} R/(\varphi_d) & \text{if } n \equiv 0 \text{ or } -1 \pmod{d} \\ 0 & \text{otherwise,} \end{cases}$$

where the non vanishing term is in degree $(d - 2)k$ if $n = dk$ or $n = dk - 1$.

Recall that, if G is a graph with vertex set VG , an *independent set* of G is a subset of VG consisting of pairwise non-adjacent vertices. Also, the *independence complex* $\text{Ind}(G)$ of G is the abstract simplicial complex with VG as its set of vertices and simplices given by all the non-empty independent sets of G . Thus $\text{Ind}(G)$ is the clique complex of the complement graph of G . Conventionally, the simplicial complex $\text{Ind}(G)$ does not contain the empty simplex, and the dimension of a simplex $\sigma \in \text{Ind}(G)$ is given by $|\sigma| - 1$. The homotopy type of $\text{Ind}(A_n)$ was computed in [Koz07] by means of discrete Morse theory, and the result is the following.

Proposition 6.3.2 ([Koz07, Proposition 11.16]). We have

$$\text{Ind}(A_n) \simeq \begin{cases} S^{k-1} & \text{if } n = 3k \text{ or } n = 3k - 1 \\ \{\text{pt}\} & \text{if } n = 3k + 1. \end{cases}$$

By comparing this with Theorem 6.3.1 we obtain the following relation between the homology of the independence complex of A_n and the φ_3 -primary component of the local homology of Br_{n+1} .

Corollary 6.3.3. We have

$$H_*(\text{Br}_{n+1}; R)_{\varphi_3} \cong \tilde{H}_{*-2} \left(\text{Ind}(A_{n-3}); \frac{R}{(\varphi_3)} \right),$$

where the homology on the left is with local coefficients, and on the right it is with trivial coefficients. \square

In general, following [Sal15], we can define the r -independence complex of a graph G as

$$\text{Ind}_r(G) = \{ \text{full subgraphs } G' \subseteq G \text{ such that each connected component of } G' \text{ has at most } r \text{ vertices} \}.$$

So $\text{Ind}_r(G)$ is an abstract simplicial complex on the set of vertices VG of the graph G , and it coincides with $\text{Ind}(G)$ for $r = 1$. The case $r = 0$ also makes sense: $\text{Ind}_0(G) = \emptyset$ for any graph G . We are going to prove the following generalization of Corollary 6.3.3.

Theorem 6.3.4. We have

$$H_*(\text{Br}_{n+1}; R)_{\varphi_d} \cong \tilde{H}_{*-d+1} \left(\text{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)} \right),$$

where the homology on the left is with local coefficients, and on the right it is with trivial coefficients.

From the expression of the φ_d -primary component of the local homology of the braid group (Theorem 6.3.1) we obtain the following consequence.

Corollary 6.3.5. We have

$$\tilde{H}_*(\text{Ind}_{d-2}(A_n)) \cong \begin{cases} \tilde{H}_*(S^{dk-2k-1}) & \text{for } n = dk \text{ or } n = dk - 1 \\ 0 & \text{otherwise.} \end{cases}$$

The Poincaré polynomial of a Coxeter group of type A_k is given by

$$W_{A_k}(q) = [k+1]_q! \quad \text{where} \quad [k]_q = \frac{q^k - 1}{q - 1} = \prod_{\substack{d|k \\ d \geq 2}} \varphi_d$$

(see for example [BB06]). Consider now a simplex $\sigma \subseteq S \cong \{1, \dots, n\}$. Denote by $\Gamma(\sigma)$ the subgraph of A_n induced by σ . Denote by $\Gamma_1(\sigma), \Gamma_2(\sigma), \dots, \Gamma_m(\sigma)$ the connected components of $\Gamma(\sigma)$, and by n_1, n_2, \dots, n_m their cardinalities

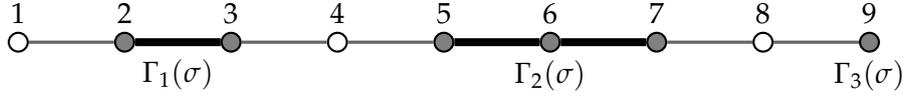


Figure 6.1: An example with $n = 9$ and $\sigma = \{2, 3, 5, 6, 7, 9\} \in \text{Ind}_3(A_9)$. In this case we have $|\Gamma_1(\sigma)| = 2$, $|\Gamma_2(\sigma)| = 3$, and $|\Gamma_3(\sigma)| = 1$.

(see Figure 6.1). The i -th connected component is a Coxeter graph of type A_{n_i} , so the entire Coxeter graph induced by σ has Poincaré polynomial

$$\begin{aligned} W_\sigma(q) &= [n_1 + 1]_q! \cdot [n_2 + 1]_q! \cdots [n_m + 1]_q! \\ &= \prod_{i=1}^m \prod_{d \geq 2} \varphi_d^{\lfloor \frac{n_i+1}{d} \rfloor} \\ &= \prod_{d \geq 2} \varphi_d^{\sum_{i=1}^m \lfloor \frac{n_i+1}{d} \rfloor}. \end{aligned}$$

Then we are interested in the homology of the weighted complex $(L_*)_\varphi$ associated to the weighted sheaf (K, R, w_φ) , where $\varphi = \varphi_d$ is a cyclotomic polynomial (with $d \geq 2$) and

$$w_\varphi(\sigma) = \varphi^{v_\varphi(\sigma)}, \quad v_\varphi(\sigma) = \sum_{i=1}^m \left\lfloor \frac{n_i+1}{d} \right\rfloor.$$

Notice that only the connected components with at least $d - 1$ vertices contribute to the φ -weight. Therefore, the weighted complex $(L_*)_\varphi$ is generated by the subgraphs having at least one component with at least $d - 1$ vertices.

Theorem 6.3.6. There is a weighted acyclic matching \mathcal{M} on K such that the set of critical simplices is given by

$$\begin{aligned} \text{Cr}(\mathcal{M}) &= \{\sigma \mid \Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_{m-1} \sqcup A_{d-1}, |\Gamma_i| \leq d-2\} \cup \text{Ind}_{d-2}(A_n) \\ &= \{\tau \sqcup A_{d-1} \mid \tau \in \text{Ind}_{d-2}(A_{n-d})\} \cup \text{Ind}_{d-2}(A_n), \end{aligned}$$

where A_{d-1} is the linear graph on the vertices $n - d + 2, \dots, n$.

Proof. First notice that removing the d -th vertex from an A_k component leaves the φ -weight unchanged:

$$v_\varphi(A_k) = \left\lfloor \frac{k+1}{d} \right\rfloor = 1 + \left\lfloor \frac{k+1-d}{d} \right\rfloor = v_\varphi(A_{d-1} \sqcup A_{k-d})$$

(see Figure 6.2).

Let $K_0 = \text{Ind}_{d-2}(A_n) \subseteq K$. This is the set of the simplices $\sigma \in K$ such that all the connected components of the induced subgraph $\Gamma(\sigma)$ have cardinality

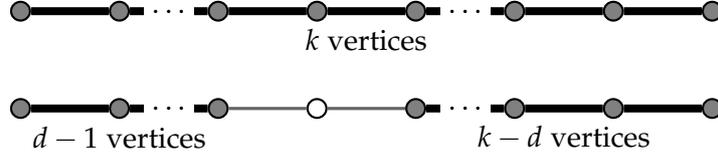


Figure 6.2: The φ -weight of an A_k component remains the same if we remove the d -th vertex, splitting A_k into $A_{d-1} \sqcup A_{k-d}$.

at most $d - 2$. Let us define on $K' = K \setminus K_0$ the following matching \mathcal{M} . For $\sigma \in K'$, with $\Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$, set

$$i(\sigma) = \min\{i : |\Gamma_i| \geq d - 1\}.$$

Then match σ with the simplex τ obtained by adding or removing from σ the d -th vertex of the component $\Gamma_{i(\sigma)}$. By the remark at the beginning of the proof, \mathcal{M} is a weighted matching. We prove that it is acyclic. In fact, suppose that an alternating path contains some subpath

$$\tau \lessdot \sigma \gtrdot \tau' \lessdot \sigma'$$

with $\tau \neq \tau'$. Then either $i(\tau) < i(\tau')$, or $i(\tau) = i(\tau') = j$ and

$$\Gamma_j(\tau) = \{a, a + 1, \dots, a + d - 2\}, \quad \Gamma_j(\tau') = \{a + 1, \dots, a + d - 1\}$$

for some $a \leq n - d + 1$. In this case the first vertex of $\Gamma_j(\tau')$ is greater than the first vertex of $\Gamma_j(\tau)$. Therefore an alternating path in K' cannot be closed.

The set of critical elements of \mathcal{M} in K' is given by

$$\text{Cr}(\mathcal{M}) = \{\sigma \mid \Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_{m-1} \sqcup A_{d-1}, |\Gamma_i| \leq d - 2\},$$

where A_{d-1} is as in the statement of the theorem. \square

Proof of Theorem 6.3.4. Consider the matching \mathcal{M} of Theorem 6.3.6. Since $K_0 = \text{Ind}_{d-2}(A_n)$ does not contribute to the weighted complex $(L_*)_\varphi$, we concentrate on the poset $K' = K \setminus K_0$. Notice that there are no non-trivial alternating paths between critical elements of K' . Therefore the boundaries between simplices in

$$\text{Cr}'(\mathcal{M}) = \{\tau \sqcup A_{d-1} \mid \tau \in \text{Ind}_{d-2}(A_{n-d})\}$$

are the same as the boundaries of $\text{Ind}_{d-2}(A_{n-d})$. In the long exact sequence (6.1), the algebraic complex C_*^0 has vanishing homology in all degrees (because

K is the full simplicial complex on n vertices), so we have an isomorphism $H_{*+1}(L_*) \cong H_*(C_*)$. Therefore

$$\begin{aligned} H_*(\text{Br}_{n+1}; R)_{\varphi_d} &\cong H_*(C_*)_{\varphi_d} \cong H_{*+1}(L_*)_{\varphi_d} \\ &\cong \tilde{H}_{*-d+1} \left(\text{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)} \right). \end{aligned}$$

In the last isomorphism there is a $(d-1)$ -shift in degree due to the loss of $d-1$ vertices when passing from $\text{Cr}'(\mathcal{M})$ to $\text{Ind}_{d-2}(A_{n-d})$; there is then a further 1-shift in degree since in L_* a simplex σ has dimension $|\sigma|$, and in $\text{Ind}_{d-2}(A_{n-d})$ it has dimension $|\sigma| - 1$; finally, it is necessary to pass to reduced homology because the empty simplex is missing in $\text{Ind}_{d-2}(A_{n-d})$. \square

For the sake of completeness, we also determine the homotopy type of the r -independence complex $\text{Ind}_{d-2}(A_n)$, obtaining again Corollary 6.3.5 as a consequence. This is a straightforward generalization of the case $d=3$ proved in [Koz07, Proposition 11.16].

Proposition 6.3.7. We have

$$\text{Ind}_{d-2}(A_n) \simeq \begin{cases} \mathcal{S}^{dk-2k-1} & \text{if } n = dk \text{ or } n = dk - 1 \\ \{\text{pt}\} & \text{otherwise.} \end{cases}$$

Proof. Let $K_0 = \text{Ind}_{d-2}(A_n)$, and let $n = qd + r$ be the euclidean division of n by d .

Let $P = \{c_d > c_{2d} > c_{3d} > \dots > c_{qd} > c_*\}$ be a linearly ordered set, where c_* is the minimum element. Define a map $f: K_0 \rightarrow P$ as follows. If $\sigma \in K_0$ does not contain any multiples of d , then set $f(\sigma) = c_*$. Otherwise, set $f(\sigma) = c_{jd}$ if $jd \in \sigma$ but $j'd \notin \sigma$ for $j' < j$. Clearly f is a poset map: if we remove a vertex from σ , then j increases or remains the same.

In $f^{-1}(c_d)$ consider the matching $\mathcal{M}_1 = \{(\sigma \setminus \{1\}, \sigma \cup \{1\}) \mid \sigma \in f^{-1}(c_d)\}$, which is justified by:

$$\sigma \in f^{-1}(c_d) \Rightarrow \{2, \dots, d-1\} \not\subseteq \sigma.$$

Similarly, in $f^{-1}(c_{jd})$ for $j \leq q$, consider the matching

$$\mathcal{M}_j = \left\{ \left(\sigma \setminus \{(j-1)d+1\}, \sigma \cup \{(j-1)d+1\} \right) \mid \sigma \in f^{-1}(c_{jd}) \right\},$$

justified by:

$$\sigma \in f^{-1}(c_{jd}) \Rightarrow \{(j-1)d+2, \dots, jd-1\} \not\subseteq \sigma.$$

Each \mathcal{M}_j is acyclic in $f^{-1}(c_{jd})$, thus

$$\mathcal{M} = \bigcup_{j=1}^q \mathcal{M}_j$$

is an acyclic matching on K_0 by the Patchwork Theorem (Theorem 1.2.6). Since each \mathcal{M}_j is a perfect matching on $f^{-1}(c_{jd})$ for $j = 1, \dots, q$, and $K_1 = f^{-1}(c_*)$ is a subcomplex of K_0 , it follows that K_0 deformation retracts onto K_1 .

Notice now that K_1 is the join of q copies of $\text{Ind}_{d-2}(A_{d-1}) \cong S^{d-3}$ and one copy of $\text{Ind}_{d-2}(A_r)$. For $r = 0$ we get

$$K_0 \cong (S^{d-3})^{*q} \cong S^{q(d-3)+q-1} = S^{qd-2q-1}$$

and for $r = d - 1$ we get

$$K_0 \cong (S^{d-3})^{*(q+1)} \cong S^{(q+1)(d-3)+q} = S^{(q+1)d-2(q+1)-1},$$

which give the first case in the corollary. If r is not 0 or $d - 1$, then $\text{Ind}_{d-2}(A_r)$ is a full simplex on r vertices, and therefore K_0 is contractible. \square

6.4 PRECISE MATCHINGS FOR ARTIN GROUPS

For a general Coxeter system (W, S) , in Section 6.2 we constructed a weighted sheaf $\mathcal{S} = (\mathcal{K}_W, R, w)$. The associated weighted complex L_* fits into the short exact sequence

$$0 \rightarrow C_* \xrightarrow{\Delta} C_*^0 \xrightarrow{\pi} L_* \rightarrow 0,$$

which gives rise to the long exact sequence (6.1):

$$\dots \xrightarrow{\pi_*} H_{k+1}(L_*) \rightarrow H_k(C_*) \xrightarrow{\Delta_*} H_k(C_*^0) \xrightarrow{\pi_*} H_k(L_*) \rightarrow \dots$$

In order to compute $H_*(C_*) = H_*(X_W; R)$, we split this long exact sequence into the short exact sequences

$$0 \rightarrow \text{coker } \pi_* \rightarrow H_m(C_*) \xrightarrow{\Delta_*} \ker \pi_* \rightarrow 0,$$

where on the left we have the cokernel of $\pi_* : H_{m+1}(C_*^0) \rightarrow H_{m+1}(L_*)$ and on the right we have the kernel of $\pi_* : H_m(C_*^0) \rightarrow H_m(L_*)$. Since $\ker \pi_*$ is a free R -module, these short exact sequences split:

$$H_m(C_*) \cong \text{coker } \pi_* \oplus \ker \pi_*.$$

Recall that the only irreducible elements of R that occur in the factorization of the weights are the cyclotomic polynomials φ_d for $d \geq 2$. As in Section 5.5, suppose from now on that we have constructed a φ -precise matching \mathcal{M}_φ for each cyclotomic polynomial $\varphi = \varphi_d$ (with $d \geq 2$). Then we have an explicit description of $\text{coker } \pi_*$ and $\ker \pi_*$ thanks to Proposition 5.5.5, and we obtain the following result.

Theorem 6.4.1 (Local homology). Under the above hypothesis, the homology of X_W with coefficients in the representation $R = \mathbb{Q}[q^{\pm 1}]$ is given by

$$H_m(X_W; R) \cong \left(\bigoplus_{\varphi} \left(\frac{R}{(\varphi)} \right)^{\oplus \text{rk } \delta_{m+1}^{\varphi}} \right) \oplus H_m(C_*^0). \quad \square$$

In particular the term $H_m(C_*^0)$ gives the free part of the homology, and the other direct summands give the torsion part. The torsion part actually takes a very particular form, as we state in the following result.

Theorem 6.4.2. Let G_W be an Artin group that admits a φ -precise matching for all cyclotomic polynomials $\varphi = \varphi_d$ (with $d \geq 2$). Then the homology $H_*(X_W; R)$ does not have φ^k -torsion for $k \geq 2$. \square

We are particularly interested in spherical and affine Artin groups. When G_W is a spherical Artin group with n generators, \mathcal{K}_W is the full simplicial complex on $S \cong \{1, \dots, n\}$, and therefore C_*^0 has a trivial homology in every dimension. Thus the formula of Theorem 6.4.1 reduces to

$$H_m(X_W; R) \cong \bigoplus_{\varphi} \left(\frac{R}{(\varphi)} \right)^{\oplus \text{rk } \delta_{m+1}^{\varphi}}.$$

When G_W is an affine Artin group with $n + 1$ generators, \mathcal{K}_W is obtained from the full simplicial complex on $S \cong \{0, 1, \dots, n\}$ by removing the top-dimensional simplex. Then we have

$$H_m(X_W; R) \cong \begin{cases} R & \text{for } m = n \\ \bigoplus_{\varphi} \left(\frac{R}{(\varphi)} \right)^{\oplus \text{rk } \delta_{m+1}^{\varphi}} & \text{for } m < n. \end{cases}$$

Remark 6.4.3 (Milnor fiber). When G_W is a spherical Artin group, the corresponding reflection arrangement of hyperplanes \mathcal{A} is finite. In this case it is well known that there is an R -module isomorphism between the local homology $H_*(X_W; R)$ and the homology with trivial coefficients $H_*(F; \mathbb{Q})$ of the Milnor fiber F of \mathcal{A} [Cal05]. The q -multiplication on the homology of X_W corresponds to the action of the monodromy operator on the homology of F . If $N = |\mathcal{A}|$, the square of the defining polynomial of the arrangement is W -invariant, thus the order of the monodromy of the Milnor fibration divides $2N$. It follows that the polynomial $q^{2N} - 1$ must annihilate the homology. Since $q^{2N} - 1$ is square-free in characteristic 0, the homology cannot have φ^k -torsion for $k \geq 2$. So the conclusion of Theorem 6.4.2 is not surprising in the case of spherical Artin groups.

The rest of this chapter is dedicated to the construction of precise matchings for all spherical and affine Artin groups. Then we obtain the following result.

Theorem 6.4.4. Every Artin group of spherical or affine type admits a φ -precise matching for each cyclotomic polynomial φ .

By Theorem 6.4.2, this has the following immediate consequence.

Corollary 6.4.5. Let G_W be an Artin group of spherical or affine type. Then the local homology $H_*(X_W; R)$ has no φ^k -torsion for $k \geq 2$. \square

The proof of Theorem 6.4.4 is split throughout the rest of this chapter. In Section 6.5 we show that it is enough to construct precise matchings in the irreducible spherical and affine cases. In the following sections we discuss all irreducible cases. For some of the infinite families $(A_n, B_n, \tilde{A}_n, \tilde{C}_n)$ we also carry out complete homology computations.

We provide a software library to construct precise matchings for any given spherical or affine Artin group. The source code is available online [Pao17b]. This library can be used to check preciseness and compute the homology.

Remark 6.4.6. Not all Artin groups admit precise matchings for every cyclotomic polynomial. For example, consider the Coxeter system (W, S) defined by the following Coxeter matrix:

$$\begin{pmatrix} 1 & 3 & 3 & 2 & \infty & 4 \\ 3 & 1 & 3 & 4 & 2 & \infty \\ 3 & 3 & 1 & \infty & 4 & 2 \\ 2 & 4 & \infty & 1 & \infty & \infty \\ \infty & 2 & 4 & \infty & 1 & \infty \\ 4 & \infty & 2 & \infty & \infty & 1 \end{pmatrix}.$$

The simplicial complex \mathcal{K}_W consists of: three 2-simplices $(\{1, 2, 4\}, \{2, 3, 5\}, \{1, 3, 6\})$, all having φ_2 -weight equal to 3; six 1-simplices with φ_2 -weight equal to 2 $(\{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{1, 6\}, \{3, 6\})$, and three 1-simplices with φ_2 -weight equal to 1 $(\{1, 2\}, \{2, 3\}, \{1, 3\})$; six 0-simplices $(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\})$, all having φ_2 -weight equal to 1; one empty simplex, with φ_2 -weight equal to 0. A φ_2 -weighted matching can only contain edges between simplices of weight 1. Since the three 1-simplices of weight 1 form a cycle, at least one of them is critical, say $\{1, 2\}$. Then the incidence number between $\{1, 2, 4\}$ and $\{1, 2\}$ is non-zero, and their φ_2 -weights differ by 2. Therefore the matching cannot be φ_2 -precise.

We introduce here a few notations that will be used in the next sections. Given a simplex $\sigma \in \mathcal{K}_W$, denote by $\Gamma(\sigma)$ the subgraph of Γ induced by σ , as in Section 6.3. The connected components of $\Gamma(\sigma)$ will be denoted by $\Gamma_i(\sigma)$ for some index i . A simplex $\sigma \subseteq S$ will be also represented as a string of bits $\epsilon_i \in \{0, 1\}$ (for $i \in S$), where $\epsilon_i = 1$ if $i \in \sigma$ and $\epsilon_i = 0$ if $i \notin \sigma$. For example, if

$S = \{1, 2, 3, 4\}$, the string representation of $\sigma = \{1, 2, 4\}$ is 1101. Finally, given a vertex $v \in S$, define

$$\sigma \underline{\vee} v = \begin{cases} \sigma \cup \{v\} & \text{if } v \notin \sigma \\ \sigma \setminus \{v\} & \text{if } v \in \sigma. \end{cases}$$

6.5 REDUCTION TO THE IRREDUCIBLE CASES

Let (W_1, S_1) and (W_2, S_2) be two Coxeter systems, and consider the product Coxeter system $(W_1 \times W_2, S_1 \sqcup S_2)$. Suppose that the Artin groups G_{W_1} and G_{W_2} admit φ -precise matchings \mathcal{M}_1 and \mathcal{M}_2 , respectively. In the following lemma we construct a φ -precise matching for $G_{W_1 \times W_2} = G_{W_1} \times G_{W_2}$.

Lemma 6.5.1. Fix a cyclotomic polynomial φ . Let \mathcal{M}_1 and \mathcal{M}_2 be φ -precise matchings on G_{W_1} and G_{W_2} , respectively. Then $G_{W_1 \times W_2}$ also admits a φ -precise matching.

Proof. First notice that the simplicial complex $\mathcal{K}_{W_1 \times W_2}$ is equal to the simplicial join $\mathcal{K}_{W_1} * \mathcal{K}_{W_2}$ of the two simplicial complexes \mathcal{K}_{W_1} and \mathcal{K}_{W_2} :

$$\mathcal{K}_{W_1 \times W_2} = \mathcal{K}_{W_1} * \mathcal{K}_{W_2} = \{\sigma_1 \sqcup \sigma_2 \mid \sigma_1 \in \mathcal{K}_{W_1}, \sigma_2 \in \mathcal{K}_{W_2}\}.$$

The weights behave well with respect to this decomposition:

$$v_\varphi(\sigma_1 \sqcup \sigma_2) = v_\varphi(\sigma_1) + v_\varphi(\sigma_2).$$

Consider on $\mathcal{K}_{W_1 \times W_2}$ the matching

$$\mathcal{M} = \left\{ \sigma_1 \sqcup \sigma_2 \rightarrow \sigma_1 \sqcup \tau_2 \mid (\sigma_2 \rightarrow \tau_2) \in \mathcal{M}_2 \right\} \cup \left\{ \sigma_1 \sqcup \sigma_2 \rightarrow \tau_1 \sqcup \sigma_2 \mid (\sigma_1 \rightarrow \tau_1) \in \mathcal{M}_1 \text{ and } \sigma_2 \text{ is critical in } \mathcal{K}_{W_2} \right\}.$$

The critical simplices $\sigma_1 \sqcup \sigma_2$ of $\mathcal{K}_{W_1 \times W_2}$ are those for which σ_1 is critical in \mathcal{K}_{W_1} and σ_2 is critical in \mathcal{K}_{W_2} .

Any alternating path in $\mathcal{K}_{W_1 \times W_2}$ projects onto an alternating path in \mathcal{K}_{W_2} via the map $\sigma_1 \sqcup \sigma_2 \mapsto \sigma_2$ (provided that multiple consecutive occurrences of the same simplex are replaced by a single occurrence). This is because an edge of the form $\sigma_1 \sqcup \sigma_2 \rightarrow \sigma_1 \sqcup \tau_2$ is in \mathcal{M} if and only if $\sigma_2 \rightarrow \tau_2$ is in \mathcal{M}_2 . The same statement is not true for \mathcal{K}_{W_1} , but we still have a weaker property which will be useful later: the projection of an alternating path to \mathcal{K}_{W_1} cannot have two consecutive edges both traversed ‘‘upwards’’ (i.e. increasing dimension). This is because if an edge of the form $\sigma_1 \sqcup \sigma_2 \rightarrow \tau_1 \sqcup \sigma_2$ is in \mathcal{M} , then $\sigma_1 \rightarrow \tau_1$ is in \mathcal{M}_1 .

Let us prove that \mathcal{M} is acyclic. Consider an alternating cycle c in $\mathcal{K}_{W_1 \times W_2}$. Its projection onto \mathcal{K}_{W_2} gives an alternating cycle, which must be trivial because \mathcal{M}_2 is acyclic. Therefore c takes the following form, for some fixed simplex $\sigma_2 \in \mathcal{K}_{W_2}$:

$$\sigma_{1,0} \sqcup \sigma_2 \succ \tau_{1,1} \sqcup \sigma_2 \prec \sigma_{1,1} \sqcup \sigma_2 \succ \cdots \succ \tau_{1,m} \sqcup \sigma_2 \prec \sigma_{1,0} \sqcup \sigma_2.$$

If σ_2 is critical in \mathcal{K}_{W_2} , then also the projection of c onto \mathcal{K}_{W_1} is an alternating cycle. By acyclicity of \mathcal{M}_1 such a projection must be the trivial cycle, so also c is trivial. On the other hand, if σ_2 is not critical, then none of the edges $\sigma_{1,i} \sqcup \sigma_2 \rightarrow \tau_{1,i} \sqcup \sigma_2$ is in \mathcal{M} , thus c must be trivial as well.

By construction, and by additivity of the weight function v_φ , the matching \mathcal{M} is φ -weighted.

Finally, suppose that $[\sigma_1 \sqcup \sigma_2 : \tau_1 \sqcup \tau_2]^{\mathcal{M}} \neq 0$, where $\sigma_1 \sqcup \sigma_2$ and $\tau_1 \sqcup \tau_2$ are critical simplices of $\mathcal{K}_{W_1 \times W_2}$ with $\dim(\sigma_1 \sqcup \sigma_2) = \dim(\tau_1 \sqcup \tau_2) + 1$. Let $\mathcal{P} \neq \emptyset$ be the set of alternating paths from $\sigma_1 \sqcup \sigma_2$ to $\tau_1 \sqcup \tau_2$. Given any path $p \in \mathcal{P}$, its projection onto \mathcal{K}_{W_2} is an alternating path from σ_2 to τ_2 .

1. Suppose $\sigma_2 = \tau_2$. Then the projected paths are trivial in \mathcal{K}_{W_2} , so \mathcal{P} is in bijection with the set of alternating paths from σ_1 to τ_1 in \mathcal{K}_{W_1} . Therefore $[\sigma_1 : \tau_1]^{\mathcal{M}_1} = \pm [\sigma_1 \sqcup \sigma_2 : \tau_1 \sqcup \tau_2]^{\mathcal{M}} \neq 0$. Since \mathcal{M}_1 is φ -precise, we conclude that $v_\varphi(\sigma_1) = v_\varphi(\tau_1) + 1$ and so that $v_\varphi(\sigma_1 \sqcup \sigma_2) = v_\varphi(\tau_1 \sqcup \tau_2) + 1$.
2. Suppose $\sigma_2 \neq \tau_2$. The projection of any $p \in \mathcal{P}$ onto \mathcal{K}_{W_2} is a non-trivial alternating path, and \mathcal{P} is non-empty, so $\dim(\sigma_2) = \dim(\tau_2) + 1$. Then $\dim(\sigma_1) = \dim(\tau_1)$. For any alternating path $p \in \mathcal{P}$, consider now its projection q onto \mathcal{K}_{W_1} . We want to prove that q is a trivial path (thus in particular $\sigma_1 = \tau_1$). Suppose by contradiction that q is non-trivial. Then, since σ_1 and τ_1 have the same dimension, one of the following three possibilities must occur.
 - The path q begins with an upward edge $\sigma_1 \prec \rho$. Then $(\rho \rightarrow \sigma_1) \in \mathcal{M}_1$, which is not possible because σ_1 is critical.
 - The path q ends with an upward edge $\rho \prec \tau_1$. Then $(\tau_1 \rightarrow \rho) \in \mathcal{M}_1$, which is not possible because τ_1 is critical.
 - The path q begins and ends with a downward edge, so it must have two consecutive upward edges somewhere in the middle. This is also not possible by previous considerations.

We proved that the projection on \mathcal{K}_{W_1} of any alternating path $p \in \mathcal{P}$ is trivial, and thus in particular $\sigma_1 = \tau_1$ (because \mathcal{P} is non-empty). Then \mathcal{P} is in bijection with the set of alternating paths from σ_2 to τ_2 in \mathcal{K}_{W_2} . We conclude as in case 1. \square

In view of this lemma, from now on we only consider irreducible Coxeter systems.

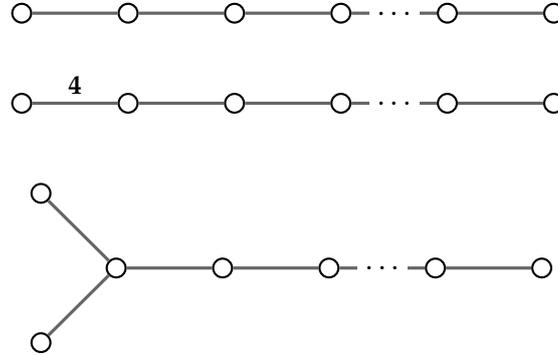


Figure 6.3: Coxeter graphs of type A_n , B_n and D_n . All these graphs have n vertices.

6.6 WEIGHT OF IRREDUCIBLE COMPONENTS

In order to compute the weight $v_\varphi(\sigma)$ of a simplex $\sigma \in \mathcal{K}_W$, we need to know the Poincaré polynomial of the parabolic subgroup W_σ of W . Let $\Gamma_1(\sigma), \dots, \Gamma_m(\sigma)$ be the connected components of the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ . Then the Poincaré polynomial of W_σ splits as a product of the Poincaré polynomials of irreducible components of spherical type:

$$W_\sigma(q) = \prod_{i=1}^m W_{\Gamma_i(\sigma)}(q), \text{ and therefore } v_\varphi(\sigma) = \sum_{i=1}^m v_\varphi(\Gamma_i(\sigma)).$$

In this section we derive formulas for the φ -weight of an irreducible component of type A_n , B_n and D_n (see Figure 6.3). This was already done in Section 6.3 for type A_n , but we summarize the argument again here.

Components of type A_n

The exponents of a Coxeter group W_{A_n} of type A_n are $1, 2, \dots, n$. Then its Poincaré polynomial is $W_{A_n}(q) = [n+1]_q!$ If φ_d is the d -th cyclotomic polynomial (for $d \geq 2$), the φ_d -weight is

$$\omega_{\varphi_d}(A_n) = \lfloor \frac{n+1}{d} \rfloor.$$

Components of type B_n

In this case the exponents are $1, 3, \dots, 2n-3, 2n-1$, and the Poincaré polynomial is $W_{B_n}(q) = [2n]_q!!$ The φ_d -weight (for $d \geq 2$) is given by

$$\omega_{\varphi_d}(B_n) = \begin{cases} \lfloor \frac{n}{d} \rfloor & \text{if } d \text{ is odd} \\ \lfloor \frac{n}{d/2} \rfloor & \text{if } d \text{ is even.} \end{cases}$$

Components of type D_n

Here the exponents are $1, 3, \dots, 2n-3, n-1$, and the Poincaré polynomial is $W_{D_n}(q) = [2n-2]_q!! \cdot [n]_q$. The φ_d -weight (for $d \geq 2$) is given by

$$\omega_{\varphi_d}(D_n) = \begin{cases} \lfloor \frac{n}{d} \rfloor & \text{if } d \text{ is odd} \\ \lfloor \frac{n-1}{d/2} \rfloor & \text{if } d \text{ is even and } d \nmid n \\ \frac{n}{d/2} & \text{if } d \text{ is even and } d \mid n. \end{cases}$$

6.7 CASE A_n

Many properties of the homology $H_*(X_W; R)$ in case A_n have been thoroughly discussed in Section 6.3. Using precise matchings we are going to obtain a new proof of the formula for the homology of braid groups (Theorem 6.3.1).

Throughout this section, let (W, S) be a Coxeter system of type A_n with generating set $S = \{1, 2, \dots, n\}$, and let $K_n^A = \mathcal{K}_W$. See Figure 2.4 for a drawing of the corresponding Coxeter graph.

For integers $f, g \geq 0$, define $K_{n,f,g}^A \subseteq K_n^A$ as follows:

$$K_{n,f,g}^A = \{\sigma \in K_n \mid \{1, 2, \dots, f\} \subseteq \sigma \text{ and } \{n-g+1, n-g+2, \dots, n\} \subseteq \sigma\}.$$

In other words, $K_{n,f,g}^A$ is the subset of K_n^A consisting of the simplices which contain the first f vertices and the last g vertices. In general, $K_{n,f,g}^A$ is not a subcomplex of K_n^A . For any $d \geq 2$, we are going to recursively construct a φ_d -weighted acyclic matching on $K_{n,f,g}^A$. The precise matchings on $K_{n,f,g}^A$ will become useful also when treating other cases (B_n, D_n , and so on). See Table 6.1 for an example.

In what follows, the notation “ $n \equiv a, \dots, b \pmod{d}$ ” means that n is congruent modulo d to some integer in the closed interval $[a, b]$.

Matching 6.7.1 (φ_d -matching on $K_{n,f,g}^A$).

- (a) If $f + g \geq n$, then $K_{n,f,g}^A$ has size at most 1, and the matching is empty. In the subsequent cases, assume $f + g < n$.
- (b) If $f \geq d$, then $K_{n,f,g}^A \cong K_{n-d,f-d,g}^A$ via removal of the first d vertices. Define the matching recursively, as in $K_{n-d,f-d,g}^A$. In the subsequent cases, assume $f \leq d-1$.
- (c) Case $n \geq d + g$.
 - (c1) If $\{1, \dots, d-1\} \subseteq \sigma$, then match σ with $\sigma \vee d$ (here the vertex d exists and can be removed, because $n \geq d + g$). Notice that for $f = d-1$ this is always the case, thus in the subsequent cases we can assume $f \leq d-2$.

Simplices		$v_\varphi(\sigma)$
	→	2
	→	2
	→	1
	(critical)	2
	(critical)	1

 Table 6.1: Matching 6.7.1 on $K_{7,1,3}^A$ for $d = 3$.

- (c2) Otherwise, if $f + 1 \in \sigma$ then match σ with $\sigma \setminus \{f + 1\}$.
- (c3) Otherwise, if $\{f + 2, \dots, d - 1\} \not\subseteq \sigma$ then match σ with $\sigma \cup \{f + 1\}$.
- (c4) We are left with the simplices σ such that $\{1, \dots, f, f + 2, \dots, d - 1\} \subseteq \sigma$ and $f + 1 \notin \sigma$. If we ignore the vertices $1, \dots, f + 1$ we are left with the simplices on the vertex set $\{f + 2, \dots, n\}$ which contain $f + 2, \dots, d - 1$; relabeling the vertices, these are the same as the simplices on the vertex set $\{1, \dots, n - f - 1\}$ which contain $1, \dots, d - 2 - f$. Then construct the matching recursively as in $K_{n-f-1, d-2-f, g}^A$.
- (d) Case $n < d + g$ (in particular, $f \leq d - 2$).
- (d1) If $n \equiv -1, 0, 1, \dots, f \pmod{d}$ and σ is either
- $$\{1, \dots, n\} \text{ or } \{1, \dots, f, f + 2, \dots, n\},$$
- then σ is critical.
- (d2) Otherwise, match σ with $\sigma \vee f + 1$.

Lemma 6.7.2. Matching 6.7.1 is acyclic.

Proof. The proof is by induction on n , the case $n = 0$ being trivial. In case (a) the matching is empty and so it is acyclic. In case (b), the matching on $K_{n-d, f-d, g}^A$ is acyclic by induction, and therefore also the matching on $K_{n, f, g}^A$ is acyclic.

Consider now case (c). Let $\eta: K_{n, f, g}^A \rightarrow \{p_1 > p_4 > p_{2,3}\}$ be the poset map that sends σ to: p_1 , if subcase (c1) applies; $p_{2,3}$, if subcase (c2) or (c3) applies; p_4 , if subcase (c4) applies. We have that $\eta(\sigma) = \eta(\tau)$ whenever σ is matched with τ . In addition, on each fiber $\eta^{-1}(p)$ the matching is acyclic (for $p = p_4$,

this follows by induction). Therefore by the Patchwork Theorem (Theorem 1.2.6) the entire matching is acyclic.

In case (d) the matching is of the form $\{\sigma \rightarrow \sigma \vee (f + 1)\}$ (possibly leaving out a pair), so it is acyclic. \square

Lemma 6.7.3. Matching 6.7.1 is φ_d -weighted.

Proof. This proof is also by induction on n . In case (b), removing the first d vertices decreases all φ_d -weights by 1, so the matching is φ_d -weighted by induction.

Consider now case (c). In (c1) we have to check that $\omega_{\varphi_d}(A_{d-1} \sqcup A_k) = \omega_{\varphi_d}(A_{d+k})$ for all $k \geq 0$. This follows immediately from the formula for φ_d -weights (Section 6.6). In (c2)-(c3) we match simplices by adding or removing $f + 1$, which alters the size of the leftmost connected component. However, this component has always size at most $d - 2$, so it does not contribute to the φ_d -weight. Finally, in (c4) the matching is φ_d -weighted by induction.

In case (d) most of the vertices belong to the rightmost connected component, which has size at least g . If we add the vertex $f + 1$ to a simplex $\sigma \in K_{n,f,g}^A$ not containing $f + 1$, we either create a leftmost connected component of size at most $d - 2$, or we join the leftmost component with the rightmost component creating the full simplex $\{1, 2, \dots, n\}$. In the former case, the leftmost component is small enough not to contribute to the φ_d -weight. In the latter case we have $\sigma = \{1, \dots, f, f + 2, \dots, n\}$; it is easy to check that $\omega_{\varphi_d}(\sigma) = \omega_{\varphi_d}(\sigma \cup \{f + 1\})$ if and only if $n \equiv f + 1, \dots, d - 2 \pmod{d}$. \square

Theorem 6.7.4 (Critical simplices in case A_n). The critical simplices of Matching 6.7.1 are given by Table 6.2. In particular, the matching is φ_d -precise for $f = g = 0$.

Remark 6.7.5. Here are a few observations about Table 6.2 and Matching 6.7.1.

- The conditions in Table 6.2 are symmetric in f and g , even if the definition of Matching 6.7.1 is not.
- The two intervals of Table 6.2 in the case $f + g < n$ are always disjoint.
- For $g = 0$, Table 6.2 simplifies a lot, as both intervals contain only one element ($d - 1$ and f_0 , respectively).
- If $f \equiv -1 \pmod{d}$ or $g \equiv -1 \pmod{d}$, then the two intervals of Table 6.2 are empty, thus there is at most 1 critical simplex.
- If $\sigma \rightarrow \tau$ is in the matching, then $\sigma = \tau \cup \{v\}$ with $v \equiv 0$ or $v \equiv f + 1 \pmod{d}$. This can be easily checked by induction.

Proof of Theorem 6.7.4. As a preliminary step, rewrite the two intervals of Table 6.2 as follows.

6. LOCAL HOMOLOGY OF ARTIN GROUPS

Case		# Critical	$ \sigma - v_{\varphi_d}(\sigma)$
$f > n$ or $g > n$		0	-
$f, g \leq n$ and $f + g \geq n$		1	$n - \lfloor \frac{n+1}{d} \rfloor$
$f + g < n$	$n \equiv \max(d - 1, f_0 + g_0 + 1), \dots, \min(f_0, g_0) + d - 1 \pmod{d}$	2	$n - \lfloor \frac{n-f}{d} \rfloor - \lfloor \frac{n-g}{d} \rfloor - 1$
	$n \equiv \max(f_0, g_0), \dots, \min(f_0 + g_0, d - 2) \pmod{d}$		$n - \lfloor \frac{n-f}{d} \rfloor - \lfloor \frac{n-g}{d} \rfloor$
	else	0	-

Table 6.2: Critical simplices of Matching 6.7.1. Here $f_0, g_0 \in \{0, \dots, d - 1\}$ are defined as $f \pmod{d}$, and $g \pmod{d}$, respectively.

- The condition $n \equiv \max(d - 1, f_0 + g_0 + 1), \dots, \min(f_0, g_0) + d - 1 \pmod{d}$ is equivalent to:

$$\begin{cases} n \equiv -1, 0, \dots, f_0 - 1 \pmod{d} \\ n - g \equiv f_0 + 1, \dots, d - 1 \pmod{d}. \end{cases} \quad (6.2)$$

- The condition $n \equiv \max(f_0, g_0), \dots, \min(f_0 + g_0, d - 2) \pmod{d}$ is equivalent to:

$$\begin{cases} n \equiv f_0, \dots, d - 2 \pmod{d} \\ n - g \equiv 0, \dots, f_0 \pmod{d}. \end{cases} \quad (6.3)$$

Throughout the proof, we will refer to these two conditions as “Condition (6.2), resp. (6.3), for $K_{n,f,g}^A$ ”.

We prove the lemma by induction on n , the case $n = 0$ being trivial. If $f + g \geq n$ there is nothing to prove, thus we can assume $f + g < n$.

Suppose to be in case (b) of Matching 6.7.1, i.e. $f \geq d$. Conditions (6.2) and (6.3) for $K_{n,f,g}^A$ are equivalent to those for $K_{n-d,f-d,g}^A$. The critical simplices of $K_{n,f,g}^A$ are in one-to-one correspondence with the critical simplices of $K_{n-d,f-d,g}^A$ via removal of the first d vertices. This correspondence decreases the size of a simplex by d , and decreases the φ_d -weight by 1. Adding $d - 1$ to the values of the last column of Table 6.2 for $K_{n-d,f-d,g}^A$ we exactly recover Table 6.2 for $K_{n,f,g}^A$.

Suppose to be in case (c), i.e. $f \leq d - 1$ and $n \geq d + g$. Critical simplices only arise from subcase (c4). Notice that $(d - 2 - f) + g < (n - f - 1)$, so

by induction the critical simplices of $K_{n-f-1, d-2-f, g}^A$ are described by the last three rows of Table 6.2. Condition (6.2) (resp. (6.3)) for $K_{n-f-1, d-2-f, g}^A$ coincides with Condition (6.3) (resp. (6.2)) for $K_{n, f, g}^A$. The critical simplices of $K_{n, f, g}^A$ are in one-to-one correspondence with the critical simplices of $K_{n-f-1, d-2-f, g}^A$ via removal of the first f vertices. This correspondence decreases the size by f , and leaves the φ_d -weight unchanged. Adding f to the values of the last column of Table 6.2 for $K_{n-f-1, d-2-f, g}^A$, we recover Table 6.2 for $K_{n, f, g}^A$.

Suppose to be in case (d), i.e., $f \leq d - 2$ and $n < d + g$. Since $f + g < n$, we must have $n - g \in \{f + 1, \dots, d - 1\}$. In particular Condition (6.3) cannot hold, and the first part of Condition (6.2) holds. Case (d1) happens if and only if $n \equiv -1, 0, \dots, f \pmod{d}$, i.e., if and only if the second part of Condition (6.2) holds. If this happens then there are two critical simplices, namely $\{1, \dots, n\}$ and $\{1, \dots, f, f + 2, \dots, n\}$. The difference $|\sigma| - v_{\varphi_d}(\sigma)$ is the same for these two simplices, and is given by $n - \lfloor \frac{n+1}{d} \rfloor = (n - 1) - \lfloor \frac{n-f}{d} \rfloor$. Since $\lfloor \frac{n-g}{d} \rfloor = 0$, we can rewrite it also as $n - \lfloor \frac{n-f}{d} \rfloor - \lfloor \frac{n-g}{d} \rfloor - 1$. \square

If we set $f = g = 0$, we obtain a precise matching on K_n^A . Then we can compute the homology $H_*(X_W; R)$ for G_W of type A_n . This gives a proof of Theorem 6.3.1.

Proof of Theorem 6.3.1 (Homology in case A_n). We simply need to apply the formula given by Theorem 6.4.1 using our precise matching on $K = K_{n, 0, 0}$. Since $f = g = 0$, there are critical simplices for $n = kd$ or $n = kd - 1$. The boundary $\delta_{m+1}^{\varphi_d}$ is non-trivial only for

$$\begin{cases} m = n - 2k = k(d - 2) & \text{if } n = kd \\ m = n - 2k + 1 = k(d - 2) & \text{if } n = kd - 1. \end{cases}$$

In both cases for $m = k(d - 2)$ we have $\text{rk } \delta_{m+1}^{\varphi_d} = 1$, and all the other boundaries are trivial. Theorem 6.3.1 follows. \square

6.8 CASE B_n

Consider a Coxeter system (W, S) of type B_n , as in Figure 6.4. In this case $K_n^B = \mathcal{K}_W$ is again the full simplex on vertices $\{1, 2, \dots, n\}$.

The situation is quite different depending on the parity of d . If d is odd, the φ_d -weight of a B_{k+1} component is equal to the φ_d -weight of an A_k component:

$$\omega_{\varphi_d}(B_{k+1}) = \left\lfloor \frac{k+1}{d} \right\rfloor = \omega_{\varphi_d}(A_k) \quad (d \text{ odd}).$$

For this reason it is possible to construct a very simple matching on K_n^B .


 Figure 6.4: Coxeter graph of type B_n .

Matching 6.8.1 (φ_d -matching on K_n^B for d odd). Match any simplex $\sigma \in K_n^B$ with $\sigma \vee 1$.

Lemma 6.8.2. Matching 6.8.1 for d odd is an acyclic weighted matching on K_n^B , with no critical simplices.

Proof. Clearly the matching is acyclic and there are no critical simplices. Let us prove that the matching is weighted. Let $\sigma \in K_n^B$. The only difference between $\Gamma(\sigma)$ and $\Gamma(\sigma \vee 1)$ is the leftmost connected component, which in one case is of type B_{k+1} and in the other case is of type A_k . Therefore $v_{\varphi_d}(\sigma) = v_{\varphi_d}(\sigma \vee 1)$. \square

Suppose from now on that d is even. The simplicial complex K_n^B is partitioned as

$$K_n^B = \bigsqcup_{q \geq 0} K_q, \quad \text{where} \quad K_q = \left\{ \sigma \in K \mid \left\lfloor \frac{|\Gamma_1(\sigma)|}{d/2} \right\rfloor = q \right\}.$$

Here $\Gamma_1(\sigma)$ is the (possibly empty) connected component of $\Gamma(\sigma)$ which contains the vertex 1. Notice that each K_q is a subset of K_n^B , but not a subcomplex in general.

Matching 6.8.3 (φ_d -matching on K_n^B for d even). For a given simplex $\sigma \in K_q$, let $|\Gamma_1(\sigma)| = q\frac{d}{2} + r$ with $0 \leq r < \frac{d}{2}$. The matching on K_q is as follows.

- (a) If $r \geq 1$ (i.e. $q\frac{d}{2} + 1 \in \sigma$), then match σ with $\sigma \setminus \left\{ q\frac{d}{2} + 1 \right\}$.
- (b) If $r = 0$ (i.e. $q\frac{d}{2} + 1 \notin \sigma$) and $\left\{ q\frac{d}{2} + 2, \dots, (q+1)\frac{d}{2} \right\} \not\subseteq \sigma$, then match σ with $\sigma \cup \left\{ q\frac{d}{2} + 1 \right\}$ (unless $n = q\frac{d}{2}$, in which case σ is critical).
- (c) We are left with the simplices σ for which neither (a) nor (b) apply, i.e. with $r = 0$ and $\left\{ q\frac{d}{2} + 2, \dots, (q+1)\frac{d}{2} \right\} \subseteq \sigma$. Ignore the first $q\frac{d}{2} + 1$ vertices and relabel the remaining ones from 1 to $n - q\frac{d}{2} - 1$, so that we are left exactly with the simplices of $K_{n - q\frac{d}{2} - 1, \frac{d}{2} - 1, 0}^A$. Then use Matching 6.7.1.

Putting together the matchings on each K_q , we obtain a matching on the full simplicial complex K_n^B .

Example 6.8.4. For $n = 4$ and $d = 4$, the simplicial complex K_n^B contains $2^4 = 16$ simplices of which 12 are matched and 4 are critical. For instance,

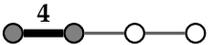
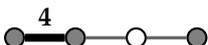
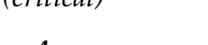
Simplices		$v_\varphi(\sigma)$	q
	(critical)	2	2
 \rightarrow 		1	1
	(critical)	1	1
 \rightarrow 		0	0
	(critical)	1	0
 \rightarrow 		0	0
 \rightarrow 		0	0
 \rightarrow 		0	0
	(critical)	0	0
 \rightarrow 		0	0

Table 6.3: Matching in case B_n with $n = 4$ and $d = 4$.

consider $\sigma = \{1, 2\} \in K_4^B$. Then $q = 1$ and $r = 0$. Since $4 \notin \sigma$, case (b) occurs. Therefore σ is matched with $\sigma \cup \{3\} = \{1, 2, 3\}$. See Table 6.3 for an explicit description of the matching in this case.

Lemma 6.8.5. Matching 6.8.3 for d even is acyclic and weighted.

Proof. Part 1: the matching is acyclic. The map $K_n^B \rightarrow (\mathbb{N}, \leq)$ which sends σ to $q = \lfloor \frac{|\Gamma_1(\sigma)|}{d/2} \rfloor$ is a poset map compatible with the matching, and its fibers are exactly the subsets K_q for $q \in \mathbb{N}$. Therefore we only need to prove that the matching on each fiber K_q is acyclic.

Let $P = \{p_c, p_{a,b}\}$ be a two-element totally ordered poset with $p_c > p_{a,b}$. For a fixed $q \in \mathbb{N}$, consider the map $\eta: K_q \rightarrow P$ which sends σ to the p_x such that σ occurs in case (x) (here cases (a) and (b) are united). Clearly η is compatible with the matching. We want to prove that it is a poset map, and for this we only need to show that if $\eta(\tau) = p_c$ and $\tau \leq \sigma$ then $\eta(\sigma) = p_c$ also. We have that $\{1, \dots, q\frac{d}{2}, q\frac{d}{2} + 2, \dots, (q+1)\frac{d}{2}\} \subseteq \tau \subseteq \sigma$. The simplex σ cannot contain the vertex $q\frac{d}{2} + 1$, because otherwise we would have $\sigma \in K_{q+1}$. Therefore $\eta(\sigma) = p_c$.

On the fiber $\eta^{-1}(p_{a,b})$ the matching is acyclic because the same vertex $q^{\frac{d}{2}} + 1$ is always added or removed. On the fiber $\eta^{-1}(p_c)$ the matching is acyclic by Lemma 6.7.2. Therefore the entire matching on K_q is acyclic.

Part 2: the matching is weighted. Let $\sigma \in K_q$ be a simplex which occurs either in case (a) or case (b). We want to show that $v_{\varphi_d}(\sigma) = v_{\varphi_d}(\sigma \vee (q^{\frac{d}{2}} + 1))$. Suppose without loss of generality that σ occurs in case (a), i.e. $r \geq 1$ and $q^{\frac{d}{2}} + 1 \in \sigma$. Let $\tau = \sigma \setminus \{q^{\frac{d}{2}} + 1\}$. The only difference between $\Gamma(\sigma)$ and $\Gamma(\tau)$ is that $\Gamma(\sigma)$ has a $B_{q^{\frac{d}{2}+r}}$ component, whereas $\Gamma(\tau)$ has a $B_{q^{\frac{d}{2}}}$ component and an A_{r-1} component instead. Since $1 \leq r < \frac{d}{2}$, we have that

$$\omega_{\varphi_d}(B_{q^{\frac{d}{2}+r}}) = \omega_{\varphi_d}(B_{q^{\frac{d}{2}}}) = q \quad \text{and} \quad \omega_{\varphi_d}(A_{r-1}) = 0$$

(the second equality holds because $r - 1 \leq \frac{d}{2} - 1 \leq d - 2$). Therefore $v_{\varphi_d}(\sigma) = v_{\varphi_d}(\tau)$.

For simplices occurring in case (c), the matching only involves changes in the connected components not containing the first vertex (i.e. connected components of type A_k). Therefore Lemma 6.7.3 applies. \square

Now we are going to describe the critical simplices. The matching has no critical simplices when n is not a multiple of $\frac{d}{2}$. On the other hand, if $n = k\frac{d}{2}$, we have two families of critical simplices: σ_q (for $0 \leq q \leq k - 2$) and σ'_q (for $0 \leq q \leq k$). See Table 6.4 and Figure 6.5 for the definition of these simplices. For instance, in Example 6.8.4, the critical simplices are: $\sigma_0 = \{2, 3, 4\}$, $\sigma'_2 = \{1, 2, 3, 4\}$, $\sigma'_1 = \{1, 2, 4\}$ and $\sigma'_0 = \{1, 3\}$. See also Table 6.5, where the critical simplices are listed by dimension.

Theorem 6.8.6 (Critical simplices in case B_n). The critical simplices for the matching on K_n^B are those defined in Table 6.4. In particular the matching is always precise, and has no critical simplices if d is odd or if d is even and $n \not\equiv 0 \pmod{\frac{d}{2}}$. In addition, if d is even and $n \equiv 0 \pmod{\frac{d}{2}}$, the incidence numbers between the critical simplices in the Morse complex are as follows:

$$\begin{aligned} [\sigma_q : \sigma_{q-1}]^{\mathcal{M}} &= (-1)^{(q-1)\frac{d}{2}} \\ [\sigma_q : \sigma'_q]^{\mathcal{M}} &= (-1)^{(k-1)(\frac{d}{2}-1)+q} \\ [\sigma'_q : \sigma_{q-2}]^{\mathcal{M}} &= (-1)^{k(\frac{d}{2}-1)+q} \\ [\sigma'_q : \sigma'_{q-1}]^{\mathcal{M}} &= (-1)^{(q-1)\frac{d}{2}}. \end{aligned}$$

Proof. **Part 1: critical simplices.** For d odd there is nothing to prove. Suppose from now on that d is even. For a given simplex $\sigma \in K_n^B$, let us consider each of the three cases that can occur in the construction of the matching.

Simplices	$ \sigma $	$v_{\varphi_d}(\sigma)$	
$\sigma_q = 1q^{\frac{d}{2}}(01^{\frac{d}{2}-1})^{k-q-2}01^{d-1}$	$n - k + q + 1$	$q + 1$	$0 \leq q \leq k - 2$
$\sigma'_q = 1q^{\frac{d}{2}}(01^{\frac{d}{2}-1})^{k-q}$	$n - k + q$	q	$0 \leq q \leq k$

Table 6.4: Description of the critical simplices for B_n , where d is even and $n = k^{\frac{d}{2}}$.

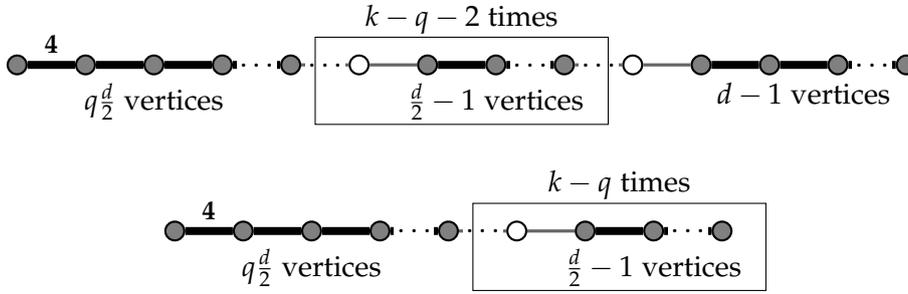


Figure 6.5: Critical simplices for B_n , where d is even and $n = k^{\frac{d}{2}}$. Above is the simplex σ_q ($0 \leq q \leq k - 2$) and below is the simplex σ'_q ($0 \leq q \leq k$).

$ \sigma $	$v_{\varphi_d}(\sigma)$	Simplices
n	k	σ'_k
$n - 1$	$k - 1$	$\sigma_{k-2}, \sigma'_{k-1}$
$n - 2$	$k - 2$	$\sigma_{k-3}, \sigma'_{k-2}$
\vdots	\vdots	\vdots
$n - k + 1$	1	σ_0, σ'_1
$n - k$	0	σ'_0

Table 6.5: Critical simplices for B_n by dimension, where d is even and $n = k^{\frac{d}{2}}$.

- (a) We have $q^{\frac{d}{2}} + 1 \in \sigma$, and σ is not critical. Indeed, none of the simplices of Table 6.4 contains the vertex $q^{\frac{d}{2}} + 1$.
- (b) We have $q^{\frac{d}{2}} + 1 \notin \sigma$ and $\{q^{\frac{d}{2}} + 2, \dots, (q+1)^{\frac{d}{2}}\} \not\subseteq \sigma$. In this case σ is critical if and only if $n = q^{\frac{d}{2}}$. Indeed, the only simplex of this type in Table 6.4 is σ'_k (which occurs for $q = k$ i.e. $n = q^{\frac{d}{2}}$).
- (c) In the remaining case, we end up with the matching on $K^A_{n-q^{\frac{d}{2}}-1, \frac{d}{2}-1, 0}$. By Theorem 6.7.4, this matching admits critical simplices if and only if

$$n - q^{\frac{d}{2}} - 1 \equiv \frac{d}{2} - 1 \pmod{d} \quad \text{or} \quad n - q^{\frac{d}{2}} - 1 \equiv -1 \pmod{d},$$

i.e. for $n \equiv 0 \pmod{\frac{d}{2}}$. Notice that if $f = \frac{d}{2} - 1$ then $d - 2 - f = \frac{d}{2} - 1$ also. Therefore, again by Theorem 6.7.4, for $n \equiv 0 \pmod{\frac{d}{2}}$ the critical simplices are exactly the ones listed in Table 6.4.

For a fixed $n = k^{\frac{d}{2}}$, the quantity $|\sigma| - v_{\varphi_d}(\sigma)$ is constant among the critical simplices and is equal to $n - k$. Thus the matching is precise.

Part 2: incidence numbers. We are going to show that there is exactly one alternating path for each of the four pairs. Notice that, if $\sigma \rightarrow \tau$ is in the matching of K_q , then $\sigma = \tau \cup \{v\}$ with $v \equiv 1 \pmod{\frac{d}{2}}$ and $v \geq q^{\frac{d}{2}} + 1$. In particular, if at a certain point of an alternating path one removes a vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$, then that vertex is never added again. But all the critical simplices contain every vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$, so any alternating path between critical simplices cannot ever drop a vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$.

Let us now consider each of the four pairs.

- (σ_q, σ_{q-1}) . We have that $\sigma_q = \sigma_{q-1} \cup \{v\}$ where $v = (q-1)^{\frac{d}{2}} + 1$, so there is the trivial alternating path $\sigma_q \rightarrow \sigma_{q-1}$ which contributes to the incidence number $[\sigma_q : \sigma_{q-1}]^{\mathcal{M}}$ by

$$[\sigma_q : \sigma_{q-1}] = (-1)^{|\{w \in \sigma_q \mid w < v\}|} = (-1)^{(q-1)^{\frac{d}{2}}}.$$

Suppose by contradiction that there exists some other (non-trivial) alternating path

$$\sigma_q \triangleright \tau_1 \triangleleft \rho_1 \triangleright \tau_2 \triangleleft \rho_2 \triangleright \dots \triangleright \tau_m \triangleleft \rho_m \triangleright \sigma_{q-1}, \quad m \geq 1.$$

Let $\sigma_q = \tau_1 \cup \{u\}$. By the previous considerations, we must have $u \equiv 1 \pmod{\frac{d}{2}}$. If $u = (k-1)^{\frac{d}{2}} + 1$ then $\tau_1 = \sigma'_k$, but this is not possible since alternating paths stop at critical simplices. Similarly, if $u = (q-1)^{\frac{d}{2}} + 1$ then we would stop at $\tau_1 = \sigma_{q-1}$. Therefore we must have $u \leq (q-2)^{\frac{d}{2}} + 1$. But then $\tau_1 \in K_{q'}$ with $q' \leq q-2$, and by induction all the subsequent simplices in the alternating path must lie in

$$\bigcup_{q'' \leq q-2} K_{q''}.$$

In particular $\sigma_{q-1} \in K_{q''}$ for some $q'' \leq q-2$, but this is a contradiction since $\sigma_{q-1} \in K_{q-1}$.

- (σ_q, σ'_q) . This case is similar to the previous one, except for the fact that $\sigma_q = \sigma'_q \cup \{v\}$ for $v = (k-1)\frac{d}{2} + 1$. So the only alternating path is the trivial one, which contributes to the incidence number by

$$\begin{aligned} [\sigma_q : \sigma'_q] &= (-1)^{|\{w \in \sigma_q \mid w < v\}|} \\ &= (-1)^{q\frac{d}{2} + (k-q-1)(\frac{d}{2}-1)} \\ &= (-1)^{(k-1)(\frac{d}{2}-1) + q}. \end{aligned}$$

- $(\sigma'_q, \sigma'_{q-1})$. This case is also similar to the previous ones. Here we have $\sigma'_q = \sigma'_{q-1} \cup \{v\}$ with $v = (q-1)\frac{d}{2} + 1$, so the contribution to the incidence number due to the trivial alternating path is

$$[\sigma'_q : \sigma'_{q-1}] = (-1)^{|\{w \in \sigma'_q \mid w < v\}|} = (-1)^{(q-1)\frac{d}{2}}.$$

- $(\sigma'_q, \sigma_{q-2})$. This case is more complicated because the only alternating path is non-trivial. Suppose we have an alternating path

$$\sigma'_q \succ \tau_1 \prec \rho_1 \succ \tau_2 \prec \rho_2 \succ \cdots \succ \tau_m \prec \rho_m \succ \sigma_{q-2}, \quad m \geq 1$$

(m must be at least 1 because σ_{q-2} is not a face of σ'_q in K). Let $\sigma'_q = \tau_1 \cup \{v\}$, with $v \equiv 1 \pmod{\frac{d}{2}}$. If $v = (q-1)\frac{d}{2} + 1$ then $\tau_1 = \sigma'_{q-1}$ and we must stop because σ'_{q-1} is already critical. If $v \leq (q-3)\frac{d}{2} + 1$ we fall into some $K_{q'}$ with $q' \leq q-3$ and it is not possible to reach $\sigma_{q-2} \in K_{q-2}$. Therefore necessarily $v = (q-2)\frac{d}{2} + 1$. Then τ_1 is matched with

$$\rho_1 = \tau_1 \cup \left\{ (q-2)\frac{d}{2} + 1 \right\}.$$

Then again, $\tau_2 = \rho_1 \setminus \{v_2\}$ and the only possibility for v_2 (in order to have $\tau_2 \neq \tau_1$ and not to fall into some $K_{q'}$ with $q' \leq q-3$) is $v_2 = (q-1)\frac{d}{2} + 1$. Proceeding by induction we obtain that

$$\begin{aligned} \tau_i &= \rho_{i-1} \setminus \left\{ (q-3+i)\frac{d}{2} + 1 \right\} \\ \rho_i &= \tau_i \cup \left\{ (q-1+i)\frac{d}{2} + 1 \right\}. \end{aligned}$$

The path stops at $\tau_{k-q+1} = \sigma_{q-2}$, and its length is $m = k - q$. See Table 6.6 for an example. The contribution of this path to the incidence number is

$$(-1)^{k-q} [\sigma'_q : \tau_1] [\rho_1 : \tau_1] \cdots [\rho_{k-q} : \tau_{k-q}] [\rho_{k-q} : \sigma_{q-2}]$$

where

$$\begin{aligned} [\rho_{i-1} : \tau_i] &= (-1)^{(q-2)\frac{d}{2}+(i-1)(\frac{d}{2}-1)} \\ [\rho_i : \tau_i] &= (-1)^{(q-2)\frac{d}{2}+(i-1)(\frac{d}{2}-1)+d-1} \\ &= (-1)^{(q-2)\frac{d}{2}+(i-1)(\frac{d}{2}-1)+1} \end{aligned}$$

(these formulas also hold for $\rho_0 = \sigma'_q$ and $\tau_{k-q+1} = \sigma_{q-2}$). Then

$$\begin{aligned} [\sigma'_q : \sigma_{q-2}]^{\mathcal{M}} &= (-1)^{k-q} \left(\prod_{i=1}^{k-q} [\rho_{i-1} : \tau_i] [\rho_i : \tau_i] \right) [\rho_{k-q} : \sigma_{q-2}] \\ &= (-1)^{k-q} (-1)^{k-q} (-1)^{(q-2)\frac{d}{2}+(k-q)(\frac{d}{2}-1)} \\ &= (-1)^{k(\frac{d}{2}-1)+q}. \end{aligned} \quad \square$$

Having a complete description of a precise matching on K_n^B , we can now compute the homology $H_*(X_W; R)$ for G_W of type B_n . We recover the result of [DCPSS99].

Theorem 6.8.7 (Homology in case B_n [DCPSS99]). For an Artin group G_W of type B_n , we have

$$H_m(X_W; R)_{\varphi_d} \cong \begin{cases} R/(\varphi_d) & \text{if } d \text{ is even, } n = k\frac{d}{2}, \text{ and } n - k \leq m \leq n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

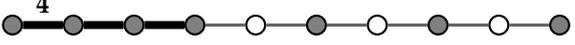
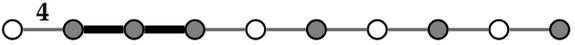
	Simplices	$v_\varphi(\sigma)$
$\sigma'_2 =$		2
$\tau_1 =$		1
$\rho_1 =$		1
$\tau_2 =$		1
$\rho_2 =$		1
$\tau_3 =$		1
$\rho_3 =$		1
$\sigma_0 =$		1

Table 6.6: The only alternating path from σ'_q to σ_{q-2} for $n = 10$, $d = 4$, $k = 5$ and $q = 2$.

Proof. For a fixed $\varphi = \varphi_d$, we need to compute the boundary δ_{m+1}^φ . Assume that $n = k\frac{d}{2}$, otherwise there are no critical simplices. In top dimension ($m+1 = n$) we have $\text{rk } \delta_n^\varphi = 1$, because $[\sigma'_k : \sigma'_{k-1}] \neq 0$. For $m \leq n - k - 1$ the boundary δ_{m+1}^φ vanishes, because there are no critical simplices in dimension $\leq n - k - 1$. For $m = n - k$ we have $\text{rk } \delta_{m+1}^\varphi = 1$, because $[\sigma'_1 : \sigma'_0] \neq 0$. Finally, for $n - k + 1 \leq m \leq n - 2$, we have (set $q = m - n + k$):

$$\begin{aligned} \text{rk } \delta_{m+1}^\varphi &= \text{rk} \begin{pmatrix} [\sigma_q : \sigma_{q-1}] & [\sigma'_{q+1} : \sigma_{q-1}] \\ [\sigma_q : \sigma'_q] & [\sigma'_{q+1} : \sigma'_q] \end{pmatrix} \\ &= \text{rk} \begin{pmatrix} (-1)^{(q-1)\frac{d}{2}} & (-1)^{k(\frac{d}{2}-1)+q+1} \\ (-1)^{(k-1)(\frac{d}{2}-1)+q} & (-1)^{q\frac{d}{2}} \end{pmatrix} = 1, \end{aligned}$$

because

$$\det \begin{pmatrix} (-1)^{(q-1)\frac{d}{2}} & (-1)^{k(\frac{d}{2}-1)+q+1} \\ (-1)^{(k-1)(\frac{d}{2}-1)+q} & (-1)^{q\frac{d}{2}} \end{pmatrix} = (-1)^{\frac{d}{2}} - (-1)^{\frac{d}{2}} = 0.$$

To summarize, we have: $\text{rk } \delta_{m+1}^\varphi = 1$ for $n - k \leq m \leq n - 1$, and $\text{rk } \delta_{m+1}^\varphi = 0$ otherwise. We conclude by applying Theorem 6.4.1. \square

6.9 CASE D_n

For $n \geq 4$ let (W, S) be a Coxeter system of type D_n , with generating set $S = \{1, 2, \dots, n\}$, and let $K_n^D = \mathcal{K}_W$ (see Figure 6.6).

We are going to construct a φ_d -precise matching on K_n^D . We split the definition according to the parity of d , and for d even we construct a matching on each

$$K_{n,g}^D = \{\sigma \in K_n^D \mid \{n - g + 1, n - g + 2, \dots, n\} \subseteq \sigma\},$$

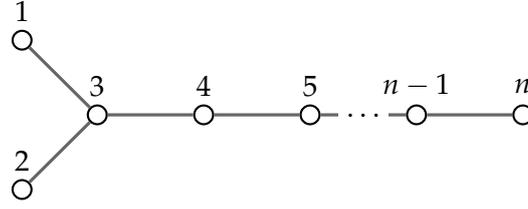
for $0 \leq g \leq n - 1$. We will need this construction to treat the cases \tilde{B}_n and \tilde{D}_n .

Matching 6.9.1 (φ_d -matching on K_n^D for d odd).

- (a) If $1 \in \sigma$ then match σ with $\sigma \vee 2$.
- (b) Otherwise, relabel the vertices $\{2, \dots, n\}$ as $\{1, \dots, n - 1\}$ and construct the matching as in K_{n-1}^A .

Matching 6.9.2 (φ_d -matching on $K_{n,g}^D$ for d even).

- (a) If $2 \notin \sigma$, relabel the vertices $\{1, 3, 4, \dots, n\}$ as $\{1, \dots, n - 1\}$ and construct the matching as in $K_{n-1,0,g}^A$.


 Figure 6.6: Coxeter graph of type D_n .

- (b) Otherwise, if $d = 2$ and $\{1, 2, 3, 4\} \not\subseteq \sigma$, proceed as follows.
- (b1) If $\{1, 2, 4\} \subseteq \sigma$: match σ with $\sigma \vee 5$ if possible (i.e. if $n - g \geq 5$); else σ is critical.
 - (b2) Otherwise: match σ with $\sigma \vee 3$ if possible (i.e. if $n - g \geq 3$); else σ is critical.
- (c) Otherwise, if $d \geq 4$ and $3 \notin \sigma$, match σ with $\sigma \vee 1$.
- (d) Otherwise, if $d = 4$ and $4 \notin \sigma$ (recall that at this point $\{2, 3\} \subseteq \sigma$), ignore the vertex 1, relabel the vertices $\{5, \dots, n\}$ as $\{1, \dots, n - 4\}$, and construct the matching as in $K_{n-4, 0, g}^A$.
- (e) Otherwise, if $d \geq 6$ and $4 \notin \sigma$, match σ with $\sigma \vee 1$.
- (f) Otherwise, if $d \geq 4$ and $1 \notin \sigma$, proceed as follows. Recall that at this point $\{2, 3, 4\} \subseteq \sigma$.
- (f1) If $\{2, \dots, \frac{d}{2} + 1\} \subseteq \sigma$, relabel the vertices $\{2, \dots, n\}$ as $\{1, \dots, n - 1\}$ and construct the matching as in $K_{n-1, \max(\frac{d}{2}, 3), g}^A$.
 - (f2) Otherwise, match σ with $\sigma \cup \{1\}$.
- (g) Otherwise, proceed as follows. Recall that at this point $\{1, 2, 3, 4\} \subseteq \sigma$. Let $k \geq 4$ be the size of the connected component $\Gamma_1(\sigma)$ of the vertex 1, in the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ . Write $k = q\frac{d}{2} + r$ where:

$$\begin{cases} 0 < r < \frac{d}{2} & \text{if } k \not\equiv 0 \pmod{\frac{d}{2}} \\ r \in \{0, \frac{d}{2}\} \text{ and } q \text{ even} & \text{if } k \equiv 0 \pmod{\frac{d}{2}}. \end{cases}$$

Define a vertex v as follows:

$$v = \begin{cases} q\frac{d}{2} + 1 & \text{if } q \text{ is even} \\ q\frac{d}{2} + 2 & \text{if } q \text{ is odd.} \end{cases}$$

It can be checked that $v = 1$ or $v \geq 5$. The idea now is that most of the times $\sigma \vee v$ has the same φ_d -weight as σ . Unfortunately there are some exceptions, so we still have to examine a few subcases.

Case	# Critical	$ \sigma - v_{\varphi_d}(\sigma)$
$n \equiv 0 \pmod{d}$	2	$n - 2 \frac{n}{d}$
$n \equiv 1 \pmod{d}$		$n - 2 \frac{n-1}{d} - 1$
else	0	-

Table 6.7: Critical simplices of Matching 6.9.1 (case D_n , d odd).

Case	Origin	$ \sigma - v_{\varphi_d}(\sigma)$
$n \equiv 0 \pmod{d}$	(a), (b) for $n = 4$ and $d = 2$, (d) for $d = 4$, (f) for $d \geq 6$ or $n = d = 4$, (g2.1), (g2.2), (g2.3)	$n - 2 \frac{n}{d}$
$n \equiv 1 \pmod{d}$	(a)	$n - 2 \frac{n-1}{d} - 1$
$n \equiv \frac{d}{2} + 1 \pmod{d}$ for $d \geq 4$	(d) for $d = 4$, (f) for $d \geq 6$, (g2.1), (g2.2), (g2.3)	$n - 2 \frac{n-1}{d}$
else	-	-

Table 6.8: Critical simplices of Matching 6.9.2 (case D_n , d even) for $g = 0$. In the second column we indicate in which parts of Matching 6.9.2 the critical simplices arise.

- (g1) Suppose $v \in \sigma$. If $v \leq n - g$, match σ with $\sigma \setminus \{v\}$. Otherwise σ is critical.
- (g2) Suppose $v \notin \sigma$. Match σ with $\sigma \cup \{v\}$, unless one of the following occurs.
- (g2.1) $v > n$ (i.e. the vertex v does not exist in S). Then σ is critical.
- (g2.2) q is even, and $\{q \frac{d}{2} + 2, \dots, (q+1) \frac{d}{2} + 1\} \subseteq \sigma$. In this case the connected components $\Gamma_1(\sigma)$ and $\Gamma_1(\sigma \cup \{v\})$ have a different φ_d -weight. Then ignore the vertices up to $q \frac{d}{2} + 1$, relabel the vertices $\{q \frac{d}{2} + 2, \dots, n\}$ as $\{1, \dots, n - q \frac{d}{2} - 1\}$ and construct the matching as in $K_{n - q \frac{d}{2} - 1, \frac{d}{2}, g}^A$.
- (g2.3) q is odd, and $\{q \frac{d}{2} + 3, \dots, (q+1) \frac{d}{2}\} \subseteq \sigma$. Similarly to case (g2.2), relabel the vertices and construct the matching as in $K_{n - q \frac{d}{2} - 2, \frac{d}{2} - 2, g}^A$.

Lemma 6.9.3. Matchings 6.9.1 and 6.9.2 are acyclic and φ_d -weighted. Critical simplices for these matchings on K_n^D are given by Tables 6.7 and 6.8. In particular, both matchings on K_n^D are φ_d -precise.

Sketch of proof. The proof is similar to those of Lemmas 6.7.2 and 6.7.3. For every $d \geq 2$, the quantity $|\sigma| - v_{\varphi_d}(\sigma)$ is constant among the critical simplices. Therefore the matchings are φ_d -precise. \square

6.10 CASE \tilde{A}_n

Consider now an affine Coxeter system (W, S) of type \tilde{A}_n (see Figure 6.7). The simplicial complex $K = \mathcal{K}_W$ consists of all the simplices $\sigma \subseteq S = \{0, \dots, n\}$ except for the full simplex $\sigma = \{0, \dots, n\}$. For any $\sigma \in K$, the induced subgraph $\Gamma(\sigma)$ consists only of connected components of type A_k .

If σ is a simplex of K , then σ misses at least one vertex of $\{0, 1, \dots, n\}$. Define h to be the first vertex missing from σ , reading counterclockwise from 0. Then σ belongs to a unique

$$K_h = \{\sigma \in K \mid h \notin \sigma \text{ and } \{0, \dots, h-1\} \subseteq \sigma\}.$$

In other words, the subsets K_h form a partition of K :

$$K = \bigsqcup_{h=0}^n K_h.$$

Matching 6.10.1. Construct a matching on a fixed K_h as follows. Let $h = qd + r$ with $0 \leq r \leq d-1$, and let $m = n - qd$. Consider the following clockwise relabeling of the vertices: $r-1$ becomes 1, $r-2$ becomes 2, \dots , 0 becomes r , n becomes $r+1$, \dots , $h+2$ becomes $m-1$, $h+1$ becomes m (the vertices $r, r+1, \dots, h$ are forgotten). This relabeling induces a poset isomorphism

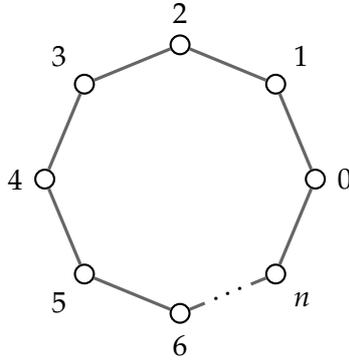
$$K_h \xrightarrow{\cong} K_{m,r,0}^A.$$

Then equip K_h with the pull-back of Matching 6.7.1 on $K_{m,r,0}^A$ (for $h = n$, the only simplex of K_h is critical). Put together the matchings on each K_h to form a matching on the entire complex K .

Lemma 6.10.2. Matching 6.10.1 is acyclic and weighted.

Proof. Consider the map $\eta: K \rightarrow (\mathbb{N}, \leq)$ which sends $\sigma \in K$ to $\min\{h \in \mathbb{N} \mid h \notin \sigma\}$. Then η is a poset map with fibers $\eta^{-1}(h) = K_h$. The matching on each $K_h \cong K_{m,r,0}^A$ is acyclic by Lemma 6.7.2, therefore the whole matching on K is acyclic.

Fix $h \in \{0, \dots, n\}$. The bijection $K_h \xrightarrow{\cong} K_{m,r,0}^A$ is such that, if $\sigma \mapsto \hat{\sigma}$, then $v_{\varphi_d}(\sigma) = v_{\varphi_d}(\hat{\sigma}) + q$. Indeed, $\Gamma(\sigma)$ is obtained from $\Gamma(\hat{\sigma})$ by adding qd vertices

Figure 6.7: Coxeter graph of type \tilde{A}_n .

to one (possibly empty) connected component, and this increases the weight by q . The matching on $K_{m,r,0}^A$ is weighted by Lemma 6.7.3, and therefore its pull-back also is. \square

Theorem 6.10.3 (Critical simplices in case \tilde{A}_n). The critical simplices of Matching 6.10.1 are those listed in Table 6.9. The only non-trivial incidence numbers between critical cells are:

$$\begin{aligned} [\tau_q : \tau'_q]^{\mathcal{M}} &= \pm 1 \quad (\text{for } n = kd + r \text{ with } 0 \leq r \leq d - 2) \\ [\sigma_{q,r} : \sigma'_{q,r}]^{\mathcal{M}} &= \pm 1 \\ [\bar{\sigma} : \sigma'_{k-1,r}]^{\mathcal{M}} &= \pm 1 \quad (\text{for } n = kd - 1). \end{aligned}$$

In particular, the matching is precise.

Proof. Part 1: critical simplices. For $n \not\equiv -1 \pmod{d}$, the matching on K_h has critical simplices only for $h \equiv n \pmod{d}$. If we write $n = kd + r$ with $0 \leq r \leq d - 2$, then $K_{qd+r} \cong K_{n-qd,r,0}^A$ has two critical simplices for $0 \leq q \leq k - 1$ and one critical simplex for $q = k$. These are the simplices τ_q and τ'_q listed in Table 6.9.

Suppose now that $n = kd - 1$. For any $h = qd + r$, with $0 \leq r \leq d - 2$, we have that $K_{qd+r} \cong K_{n-qd,r,0}^A$ has two critical simplices because $n - qd \equiv -1 \pmod{d}$. In addition, for $h = n$, we have that $K_n \cong K_{d-1,d-1,0}^A$ has one critical simplex. In the remaining cases the matching on K_h has no critical simplices. The critical simplices are therefore those named $\sigma_{q,r}$, $\sigma'_{q,r}$, and $\bar{\sigma}$ in Table 6.9.

Part 2: incidence numbers for $n = kd + r$. We want to find the incidence numbers between critical simplices of consecutive dimensions. We start with the case $n = kd + r$, with $0 \leq r \leq d - 2$. First, let us look for alternating paths

6. LOCAL HOMOLOGY OF ARTIN GROUPS

Case	Simplices	$ \sigma $	$v_{\varphi_d}(\sigma)$	
$n = kd + r$ $\not\equiv -1 \pmod{d}$	τ_q	$n - 2(k - q) + 1$	$q + 1$	$0 \leq q \leq k - 1$
	τ'_q	$n - 2(k - q)$	q	$0 \leq q \leq k$
$n = kd - 1$	$\sigma_{q,r}$	$n - 2(k - q) + 2$	$q + 1$	$0 \leq q \leq k - 2$ $0 \leq r \leq d - 2$
	$\sigma'_{q,r}$	$n - 2(k - q) + 1$	q	$0 \leq q \leq k - 1$ $0 \leq r \leq d - 2$
	$\bar{\sigma} = 1^n 0$	n	k	

Simplices

$$\begin{aligned} \tau_q &= 1^{qd+r} 0 1^{d-1} 0 (1^r 0 1^{d-2-r} 0)^{k-q-1} \\ \tau'_q &= 1^{qd+r} 0 (1^r 0 1^{d-2-r} 0)^{k-q} \\ \sigma_{q,r} &= 1^{qd+r} 0 1^{d-1} 0 (1^{d-2-r} 0 1^r 0)^{k-q-2} 1^{d-2-r} 0 \\ \sigma'_{q,r} &= 1^{qd+r} 0 (1^{d-2-r} 0 1^r 0)^{k-q-1} 1^{d-2-r} 0 \end{aligned}$$

Table 6.9: Description of the critical simplices for \tilde{A}_n . Below, the binary string notation is used.

from τ_q to τ'_q (for $0 \leq q \leq k - 1$). Set $h = qd + r$. Suppose we have one such path:

$$\tau_q \succ \zeta_1 \prec \rho_1 \succ \zeta_2 \prec \rho_2 \succ \cdots \succ \zeta_m \prec \rho_m \succ \tau'_q.$$

If at some point a vertex $v \in \{0, \dots, h - 1\}$ is removed, then the path falls into some $K_{h'}$ with $h' < h$ and can never return in K_h . Therefore the path must be entirely contained in $K_h \cong K_{n-qd, r}^A$, and by Theorem 6.7.4 it must be the trivial path $\tau_q \succ \tau'_q$. Thus $[\tau_q : \tau'_q] = \pm 1$.

The other pairs of critical simplices in consecutive dimensions are (τ'_{q+1}, τ_q) for $0 \leq q \leq k - 1$. There is a trivial path $\tau'_{q+1} \succ \tau_q$ which consists in removing the vertex $h = qd + r$, and contributes to the incidence number by

$$[\tau'_{q+1} : \tau_q] = (-1)^h.$$

Suppose we have some other (non-trivial) alternating path:

$$\tau'_{q+1} \succ \zeta_1 \prec \rho_1 \succ \zeta_2 \prec \rho_2 \succ \cdots \succ \zeta_m \prec \rho_m \succ \tau_q, \quad m \geq 1. \quad (6.4)$$

Let $\rho_m = \tau_q \cup \{v\}$. If $v = h$ then $\rho_m = \tau'_{q+1}$, which is excluded. Then v must be one of the other vertices that do not belong to τ_q . They are of the

form $v = n - sd$ (for $0 \leq s \leq k - q - 1$) or $v = n - (d - 1 - r) - sd$ (for $0 \leq s \leq k - q - 2$). If $v = n - (d - 1 - r) - sd$, then ρ_m is matched with $\rho_m \cup \{n - (s + 1)d\}$. This is impossible because ρ_m must be matched with some $\zeta_m \triangleleft \rho_m$. Similarly, if $v = n - sd$ with $0 \leq s \leq k - q - 2$, then ρ_m is matched with $\rho_m \cup \{n - (d - 1 - r) - sd\}$ and not with some $\zeta_m \triangleleft \rho_m$. The only remaining possibility is $v = n - sd$ with $s = k - q - 1$, i.e. $v = n - (k - q - 1)d = (q + 1)d + r$. In this case ρ_m is matched with $\zeta_m = \rho_m \setminus \{qd + 2r + 1\}$. Going on with the same argument, we have exactly one way to continue the alternating path (from right to left in (6.4)), and we eventually end up with τ'_{q+1} . From left to right, the obtained alternating path is the following:

$$\begin{aligned}
\zeta_1 &= \tau'_{q+1} \setminus \{qd + 2r + 1\} \\
\rho_1 &= \zeta_1 \cup \{n\} \\
\zeta_2 &= \rho_1 \setminus \{qd + r\} \\
\rho_2 &= \zeta_2 \cup \{n - (d - 1 - r)\} \\
\zeta_3 &= \rho_2 \setminus \{n\} \\
\rho_3 &= \zeta_3 \cup \{n - d\} \\
\zeta_4 &= \rho_3 \setminus \{n - (d - 1 - r)\} \\
&\vdots \\
\zeta_m &= \rho_{m-1} \setminus \{(q + 1)d + 2r + 1\} \\
\rho_m &= \zeta_m \cup \{qd + 2r + 1\} \\
\tau_q &= \rho_m \setminus \{(q + 1)d + r\}.
\end{aligned}$$

The length of the path is $m = 2(k - q)$. Apart from $qd + r$, which is the vertex in $\tau'_{q+1} \setminus \tau_q$, the other vertices are added and removed exactly once during the path. If a certain v is added in some ρ_i and removed in some ζ_j (the removal might possibly come before the addition), then

$$[\rho_i : \zeta_i][\rho_{j-1} : \zeta_j] = \begin{cases} 1 & \text{if } v = n \\ -1 & \text{otherwise.} \end{cases}$$

This is true because, except for $v = n$, between the addition and the removal of v exactly one vertex u with $u < v$ has been added/removed. Namely, between the addition and the removal of a vertex $v = jd + 2r + 1$ (for $q \leq j \leq k - 1$) the vertex $u = jd + r$ is added/removed, and between the addition of the removal of a vertex $v = jd + r$ (for $q + 1 \leq j \leq k - 1$) the vertex $u = (j - 1)d + 2r + 1$ is added. Therefore the alternating path (6.4) contributes to the incidence number by

$$(-1)^m \cdot (-1)^{m-1} \cdot [\zeta_2 : \rho_1] = (-1)^{qd+r+1} = (-1)^{h+1}.$$

Finally, the incidence number is given by

$$[\tau'_{q+1} : \tau_q]^{\mathcal{M}} = (-1)^h + (-1)^{h+1} = 0.$$

Part 3: incidence numbers for $n = kd - 1$. Consider a generic alternating path starting from $\bar{\sigma} = \{0, \dots, n - 1\}$:

$$\bar{\sigma} \succ \zeta_1 \triangleleft \rho_1 \succ \zeta_2 \triangleleft \rho_2 \succ \cdots \succ \zeta_m \triangleleft \rho_m \succ \zeta_{m+1}.$$

Let $\bar{\sigma} = \zeta_1 \cup \{v\}$. If $n - d + 1 \leq v \leq n - 1$ then $\zeta_1 = \sigma'_{k-1,r}$ for $r = n - 1 - v$. Therefore we have trivial alternating paths from $\bar{\sigma}$ to any of the $\sigma'_{k-1,r}$. If $v \leq n - d$ then $\zeta_1 \in K_h$ with $h \leq n - d = (k - 1)d - 1$. None of these K_h 's contains critical simplices ζ_{m+1} with $|\zeta_{m+1}| = n - 1$, and the alternating path cannot return in any $K_{h'}$ with $h' \geq (k - 1)d$. Thus there are no other alternating paths from $\bar{\sigma}$ to critical simplices of K . Then the non-trivial incidence numbers involving $\bar{\sigma}$ are:

$$[\bar{\sigma} : \sigma'_{k-1,r}]^M = \pm 1.$$

Consider now a generic alternating path from σ_{q,r_1} to σ'_{q,r_2} :

$$\sigma_{q,r_1} \succ \zeta_1 \triangleleft \rho_1 \succ \zeta_2 \triangleleft \rho_2 \succ \cdots \succ \zeta_m \triangleleft \rho_m \succ \sigma'_{q,r_2}.$$

Let $\rho_m = \sigma'_{q,r_2} \cup \{v\}$. Adding v to σ'_{q,r_2} causes the creation of a connected component Γ_i with $|\Gamma_i| \equiv -1 \pmod{d}$. This means that ρ_m is matched with a simplex of higher dimension, or is not matched at all (this happens for $v = (q + 1)d - 1$). Therefore the alternating path must be trivial, and it occurs only for $r_1 = r_2$. Then the non-trivial incidence numbers of the form $[\sigma_{q,r_1} : \sigma'_{q,r_2}]^M$ are:

$$[\sigma_{q,r} : \sigma'_{q,r}]^M = \pm 1.$$

Finally consider a generic alternating path from σ'_{q+1,r_1} to σ_{q,r_2} :

$$\sigma'_{q+1,r_1} \succ \zeta_1 \triangleleft \rho_1 \succ \zeta_2 \triangleleft \rho_2 \succ \cdots \succ \zeta_m \triangleleft \rho_m \succ \sigma_{q,r_2}. \quad (6.5)$$

As before, we work backwards. Let $\rho_m = \sigma_{q,r_2} \cup \{v\}$. Apart from the choices $v = qd + r$ and $v = (q + 1)d + r$, in all other cases ρ_m has a connected component of size $\equiv -1 \pmod{d}$ and this prevents the continuation of the alternating path. For $v = qd + r$ we obtain $\rho_m = \sigma'_{q+1,r_2}$, so we have a trivial alternating path from σ'_{q+1,r_2} to σ_{q,r_2} . For $v = (q + 1)d + r$, iterating the same argument, we have exactly one way to continue the alternating path (from right to left in (6.5)) and we end up with σ'_{q+1,r_2} . From left to right, the path is as follows (set $r = r_2$):

$$\begin{aligned} \zeta_1 &= \sigma'_{q+1,r} \setminus \{(q + 1)d - 1\} \\ \rho_1 &= \zeta_1 \cup \{n\} \\ \zeta_2 &= \rho_1 \setminus \{qd + r\} \\ \rho_2 &= \zeta_2 \cup \{n - (d - 1 - r)\} \\ \zeta_3 &= \rho_2 \setminus \{n\} \\ \rho_3 &= \zeta_3 \cup \{n - d\} \end{aligned}$$

$$\begin{aligned}\zeta_4 &= \rho_3 \setminus \{n - (d - 1 - r)\} \\ &\vdots \\ \zeta_m &= \rho_{m-1} \setminus \{(q + 2)d - 1\} \\ \rho_m &= \zeta_m \cup \{(q + 1)d - 1\} \\ \sigma_{q,r} &= \rho_m \setminus \{(q + 1)d + r\}.\end{aligned}$$

The length of the path is $m = 2(k - q) - 1$. As happened in the case $n = kd + r$, the contribution of this alternating path to the incidence number is given by

$$(-1)^m \cdot (-1)^{m-1} \cdot [\zeta_2 : \rho_1] = (-1)^{qd+r+1}.$$

The contribution of the trivial path $\sigma'_{q+1,r} \succ \sigma_{q,r}$ is given by $(-1)^{qd+r}$. Therefore

$$[\sigma'_{q+1,r} : \sigma_{q,r}]^{\mathcal{M}} = (-1)^{qd+r} + (-1)^{qd+r+1} = 0. \quad \square$$

We are now able to recover the result of [CMS08b] about the homology $H_*(X_W; R)$ when G_W is an Artin group of type \tilde{A}_n .

Theorem 6.10.4 (Homology in case \tilde{A}_n [CMS08b]). For an Artin group G_W of type \tilde{A}_n , we have

$$H_m(X_W; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus d-1} & \text{if } n = kd - 1 \text{ and } m = n - 2i + 1 \ (1 \leq i \leq k) \\ R/(\varphi_d) & \text{if } n = kd + r \text{ and } m = n - 2i \ (1 \leq i \leq k) \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq r \leq d - 2$ in the second case.

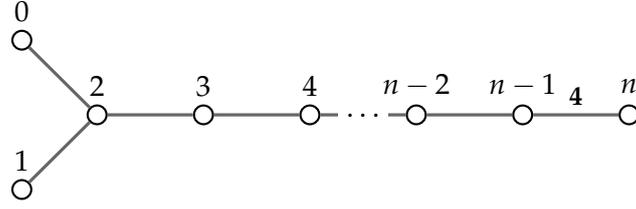
Proof. We apply Theorems 6.4.1 and 6.10.3. For $n = kd + r$ ($0 \leq r \leq d - 2$), the boundary map $\delta_{m+1}^{\varphi_d}$ has rank 1 when $m + 1 = n - 2(k - q) + 1$ (for $0 \leq q \leq k - 1$, due to τ_q); it has rank 0 otherwise. For $n = kd - 1$, the boundary map $\delta_{m+1}^{\varphi_d}$ has rank $d - 1$ if $m + 1 = n - 2(k - q) + 2$ (for $0 \leq q \leq k - 2$, due to the simplices $\sigma_{q,r}$) or if $m + 1 = n$ (due to $\bar{\sigma}$); it has rank 0 otherwise. \square

6.11 CASE \tilde{B}_n

Consider now, for $n \geq 3$, an affine Coxeter system (W, S) of type \tilde{B}_n (see Figure 6.8). Throughout this section, let $K = \mathcal{K}_W$. The matching is very simple for d odd, and has exactly one critical simplex. For d even, the situation is more complicated.

Matching 6.11.1 (φ_d -matching for d odd). For $\sigma \neq \{1, 2, \dots, n\}$, match σ with $\sigma \vee 0$. Then $\{1, 2, \dots, n\}$ is the only critical simplex.

Matching 6.11.2 (φ_d -matching for d even). For $\sigma \in K$, let k be the size of the connected component $\Gamma_n(\sigma)$ of the vertex n , in the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ . Let $k = q\frac{d}{2} + r$, with $0 \leq r < \frac{d}{2}$.


 Figure 6.8: Coxeter graph of type \tilde{B}_n .

- (a) If $r \geq 1$, match σ with $\sigma \vee (n - q^{\frac{d}{2}})$, unless $\sigma = \{0, 2, 3, \dots, n\}$ and $r = 1$ (in this case σ is critical).
- (b) If $r = 0$ and $\{n - (q + 1)^{\frac{d}{2}} + 1, \dots, n - q^{\frac{d}{2}} - 1\} \subseteq \sigma$, ignore vertices $\geq n - q^{\frac{d}{2}}$, relabel vertices $\{0, 1, \dots, n - q^{\frac{d}{2}} - 1\}$ as $\{1, 2, \dots, n - q^{\frac{d}{2}}\}$, and construct the matching as in $K_{n - q^{\frac{d}{2}}, \frac{d}{2} - 1}^D$.
- (c) If $r = 0$ and $\{n - (q + 1)^{\frac{d}{2}} + 1, \dots, n - q^{\frac{d}{2}} - 1\} \not\subseteq \sigma$, proceed as follows.
- (c1) If $|\sigma| = n$ (i.e. σ is either $\{0, 2, 3, \dots, n\}$ or $\{1, 2, 3, \dots, n\}$), then σ is critical.
- (c2) If $n = (q + 1)^{\frac{d}{2}}$ and $\sigma = \{0, 2, 3, \dots, n - q^{\frac{d}{2}} - 1, n - q^{\frac{d}{2}} + 1, \dots, n\}$, then σ is critical.
- (c3) Otherwise, match σ with $\sigma \vee (n - q^{\frac{d}{2}})$.

Lemma 6.11.3. Matchings 6.11.1 and 6.11.2 are acyclic and φ_d -weighted. For d odd, Matching 6.11.1 has exactly one critical simplex σ which satisfies $|\sigma| - v_{\varphi_d}(\sigma) = n - \lfloor \frac{n}{d} \rfloor$. For d even, all critical simplices σ of Matching 6.11.2 satisfy $|\sigma| - v_{\varphi_d}(\sigma) = n - \lfloor \frac{n}{d/2} \rfloor$. In particular, both matchings are φ_d -precise.

Sketch of proof. The first part can be proved along the lines of Lemmas 6.7.2 and 6.7.3. The formulas for $|\sigma| - v_{\varphi_d}(\sigma)$ can be checked by examining all critical simplices. Finally, since the quantity $|\sigma| - v_{\varphi_d}(\sigma)$ is constant among the critical simplices, both matchings are φ_d -precise. \square

6.12 CASE \tilde{C}_n

Let (W, S) be a Coxeter system of type \tilde{C}_n , with generating set $S = \{0, 1, \dots, n\}$, and let $K = \mathcal{K}_W$. The corresponding Coxeter graph Γ is shown in Figure 6.9. For any $\sigma \in K$, the subgraph $\Gamma(\sigma)$ of Γ splits as a union of connected components of type B_k (those containing the first or the last vertex) and of type A_k (the remaining ones).

Recall that K_m^B is the full simplicial complex on $\{1, \dots, m\}$, endowed with the weight function of B_m (see Section 6.8). We are going to construct a precise matching for the case \tilde{C}_n using the precise matching on K_m^B for $1 \leq m \leq n$.

Figure 6.9: Coxeter graph of type \tilde{C}_n .

For $h \in \{0, \dots, n\}$ set

$$K_h = \{\sigma \in K \mid h \notin \sigma \text{ and } \{0, \dots, h-1\} \subseteq \sigma\}.$$

As in the \tilde{A}_n case, since every simplex $\sigma \in K$ misses at least one vertex, the subsets K_h form a partition of K :

$$K = \bigsqcup_{h=0}^n K_h.$$

Matching 6.12.1. Construct a matching on a fixed K_h as follows. Ignore the vertices $0, \dots, h$ and relabel the remaining ones from right to left: n becomes 1, $n-1$ becomes 2, \dots , $h+1$ becomes $n-h$. This induces a poset isomorphism

$$K_h \xrightarrow{\cong} K_{n-h}^B.$$

Then equip K_h with the pull-back of the matching on K_{n-h}^B constructed in Section 6.8 (for $h = n$, the only simplex of K_h is critical). The matching on the entire complex K is obtained as the union of the matchings on each K_h .

Remark 6.12.2. For d odd we are simply matching σ with $\sigma \vee n$ for all σ except $\sigma = \{0, \dots, n-1\}$.

Lemma 6.12.3. Matching 6.12.1 is acyclic and weighted.

Proof. As in the proof of Lemma 6.10.2, the subsets K_h are the fibers of a poset map, and the matching is acyclic on each $K_h \cong K_{n-h}^B$ by Lemmas 6.8.2 and 6.8.5.

If a simplex $\sigma \in K_h$ is sent to $\hat{\sigma}$ by the isomorphism $K_h \xrightarrow{\cong} K_{n-h}^B$, then $v_{\varphi_d}(\sigma) = v_{\varphi_d}(\hat{\sigma}) + \omega_{\varphi_d}(B_h)$. The matching on K_{n-h}^B is weighted, and therefore the matching on K_h also is. \square

6. LOCAL HOMOLOGY OF ARTIN GROUPS

Case	Simplices	$ \sigma $	$v_{\varphi_d}(\sigma)$	
d odd	$\bar{\sigma} = 1^n 0$	n	1 or 0	
d even	σ_{q_1, q_2}	$n - k + q_1 + q_2 + 1$	$q_1 + q_2 + 1$	$0 \leq q_1 + q_2 \leq k - 2$
	σ'_{q_1, q_2}	$n - k + q_1 + q_2$	$q_1 + q_2$	$0 \leq q_1 + q_2 \leq k$

Simplices
$\sigma_{q_1, q_2} = 1^{q_1 \frac{d}{2} + r} 0 1^{d-1} 0 (1^{\frac{d}{2}-1} 0)^{k-q_1-q_2-2} 1^{q_2 \frac{d}{2}}$
$\sigma'_{q_1, q_2} = 1^{q_1 \frac{d}{2} + r} 0 (1^{\frac{d}{2}-1} 0)^{k-q_1-q_2} 1^{q_2 \frac{d}{2}}$

Table 6.10: Description of the critical simplices for \tilde{C}_n . When d is even, set $n = k \frac{d}{2} + r$.

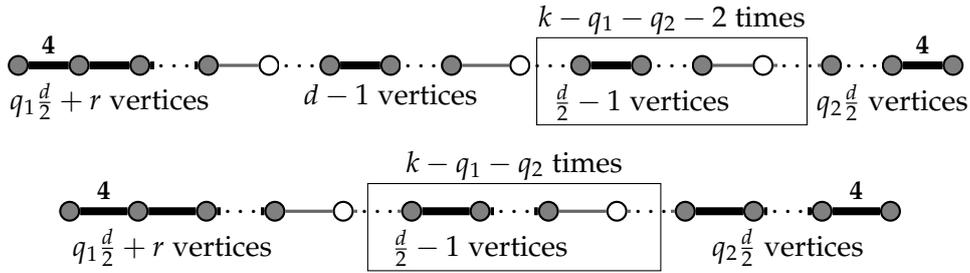


Figure 6.10: Critical simplices for \tilde{C}_n , with d even and $n = k \frac{d}{2} + r$. The diagram for the simplex σ_{q_1, q_2} is on top and the diagram for the simplex σ'_{q_1, q_2} is below it.

Simplices	
$\sigma_{l-1, 0}$	$\sigma'_{l, 0}$
$\sigma_{l-2, 1}$	$\sigma'_{l-1, 1}$
\vdots	\vdots
$\sigma_{1, l-2}$	$\sigma'_{2, l-2}$
$\sigma_{0, l-1}$	$\sigma'_{1, l-1}$
	$\sigma'_{0, l}$

Table 6.11: Critical simplices for \tilde{C}_n in dimension $m = n - k + l$ ($0 \leq l \leq k$), where d is even and $n = k \frac{d}{2} + r$. For $l = k$ only the second column occurs.

Theorem 6.12.4 (Critical simplices in case \tilde{C}_n). The critical simplices for the matching on K are those listed in Table 6.10. In particular, the matching is precise. In addition, the only non-trivial incidence numbers between critical simplices in the Morse complex are as follows (for d even and $n = k\frac{d}{2} + r$):

$$\begin{aligned} [\sigma_{q_1, q_2} : \sigma_{q_1, q_2-1}]^{\mathcal{M}} &= (-1)^\alpha \\ [\sigma_{q_1, q_2} : \sigma_{q_1-1, q_2}]^{\mathcal{M}} &= (-1)^{\beta+1} \\ [\sigma_{q_1, q_2} : \sigma'_{q_1, q_2}]^{\mathcal{M}} &= (-1)^{\beta+\frac{d}{2}+1} \\ [\sigma'_{q_1, q_2} : \sigma_{q_1, q_2-2}]^{\mathcal{M}} &= (-1)^{\beta+1} \\ [\sigma'_{q_1, q_2} : \sigma_{q_1-2, q_2}]^{\mathcal{M}} &= (-1)^\beta \\ [\sigma'_{q_1, q_2} : \sigma'_{q_1, q_2-1}]^{\mathcal{M}} &= (-1)^{\alpha+1} \\ [\sigma'_{q_1, q_2} : \sigma'_{q_1-1, q_2}]^{\mathcal{M}} &= (-1)^{\beta+\frac{d}{2}}, \end{aligned}$$

where $\alpha = (k - q_2)(\frac{d}{2} - 1) + q_1 + r + \frac{d}{2}$ and $\beta = q_1\frac{d}{2} + r$.

Proof. As we have already said, for d odd there is exactly one critical simplex. Suppose from now on that d is even. Let $n = k\frac{d}{2} + r$, with $0 \leq r \leq \frac{d}{2} - 1$.

Part 1: the critical simplices. In K_h there are critical simplices if and only if $n - h \equiv 0 \pmod{\frac{d}{2}}$, i.e. when $h = q_1\frac{d}{2} + r$ for some q_1 (with $0 \leq q_1 \leq k$). By Theorem 6.8.6 there are two families of critical simplices: σ_{q_1, q_2} (for $0 \leq q_1 + q_2 \leq k - 2$) and σ'_{q_1, q_2} (for $0 \leq q_1 + q_2 \leq k$), as shown in Table 6.10 and Figure 6.10. In a fixed dimension $m = n - k + l$ ($0 \leq l \leq k$) there are $2l + 1$ critical simplices if $l \leq k - 1$ and l critical simplices if $l = k$. See Table 6.11.

Part 2: paths ending in σ_{q_1, q_2} . Consider a generic alternating path starting from any critical cell ρ_0 and ending in a critical cell of the form σ_{q_1, q_2} :

$$\rho_0 \succ \tau_1 \prec \rho_1 \succ \tau_2 \prec \rho_2 \succ \cdots \succ \tau_m \prec \rho_m \succ \sigma_{q_1, q_2}.$$

Let $\rho_m = \sigma_{q_1, q_2} \cup \{v\}$. If $v = q_1\frac{d}{2} + r$ then $\rho_m = \sigma'_{q_1+2, q_2}$ and the alternating path stops. If $v = n - q_2\frac{d}{2}$ then the alternating path stops at $\rho_m = \sigma_{q_1, q_2+1}$. Suppose $q_1 + q_2 \leq k - 3$, otherwise there are no more cases. If $q_1\frac{d}{2} + r + d < v < n - q_2\frac{d}{2}$, then $\Gamma(\rho_m)$ has at least one connected component of size $d - 1$ and therefore ρ_m is matched with a simplex of higher dimension; thus the path stops without having reached a critical simplex. If $v = q_1\frac{d}{2} + r + d$ then ρ_m is matched with

$$\tau_m = \rho_m \setminus \left\{ q_1\frac{d}{2} + r + \frac{d}{2} \right\} = 1^{q_1\frac{d}{2}+r} 01^{\frac{d}{2}-1} 01^{d-1} 0(1^{\frac{d}{2}-1} 0)^{k-q_1-q_2-3} 01^{q_2\frac{d}{2}}$$

in the binary string notation. From here the path can continue in many ways. Let $\rho_{m-1} = \tau_m \cup \{w\}$.

- If $w = q_1\frac{d}{2} + r$, we end up with $\rho_{m-1} = \sigma_{q_1+1, q_2}$.

- If $w = q_1 \frac{d}{2} + r + \frac{d}{2}$ we go back to ρ_m .
- If $w > q_1 \frac{d}{2} + r + \frac{3}{2}d$, then $\Gamma(\rho_{m-1})$ has at least one connected component of size $d - 1$ and therefore ρ_{m-1} is matched with a simplex of a higher dimension.
- If $w = q_1 \frac{d}{2} + r + \frac{3}{2}d$, then ρ_{m-1} is matched with

$$\begin{aligned} \tau_{m-1} &= \rho_{m-1} \setminus \left\{ q_1 \frac{d}{2} + r + d \right\} \\ &= 1^{q_1 \frac{d}{2} + r} 0^{1^{\frac{d}{2}-1}} 0^{1^{\frac{d}{2}-1}} 0^{d-1} 0^{(1^{\frac{d}{2}-1})^{k-q_1-q_2-4}} 0^{1^{q_2 \frac{d}{2}}}. \end{aligned}$$

By induction, repeating the same argument as above, this path can be continued in exactly one way and it eventually arrives at the critical simplex σ'_{q_1, q_2+2} . The path has length $m = k - q_1 - q_2 - 2$ and is as follows.

$$\begin{aligned} \tau_1 &= \sigma'_{q_1, q_2+2} \setminus \left\{ n - q_2 \frac{d}{2} \right\} \\ \rho_1 &= \tau_1 \cup \left\{ n - q_2 \frac{d}{2} - d \right\} \\ \tau_2 &= \rho_1 \setminus \left\{ n - q_2 \frac{d}{2} - \frac{d}{2} \right\} \\ \rho_2 &= \tau_2 \cup \left\{ n - q_2 \frac{d}{2} - \frac{3}{2}d \right\} \\ \tau_3 &= \rho_2 \setminus \left\{ n - q_2 \frac{d}{2} - d \right\} \\ &\vdots \\ \rho_{m-1} &= \tau_{m-1} \cup \left\{ q_1 \frac{d}{2} + r + d \right\} \\ \tau_m &= \rho_{m-1} \setminus \left\{ q_1 \frac{d}{2} + r + \frac{3}{2}d \right\} \\ \rho_m &= \tau_m \cup \left\{ q_1 \frac{d}{2} + r + \frac{d}{2} \right\} \\ \sigma_{q_1, q_2} &= \rho_m \setminus \left\{ q_1 \frac{d}{2} + r + d \right\}. \end{aligned}$$

Part 3: paths ending in σ'_{q_1, q_2} . Consider now a generic alternating path starting from any critical cell ρ_0 and ending in a critical cell of the form σ'_{q_1, q_2} :

$$\rho_0 \succ \tau_1 \prec \rho_1 \succ \tau_2 \prec \rho_2 \succ \cdots \succ \tau_m \prec \rho_m \succ \sigma'_{q_1, q_2}. \quad (6.6)$$

As usual, let $\rho_m = \sigma'_{q_1, q_2} \cup \{v\}$. For the same reasons as above, there are only three possibilities: $v = q_1 \frac{d}{2} + r$, $v = q_1 \frac{d}{2} + r + \frac{d}{2}$, $v = n - q_2 \frac{d}{2}$; in all the other cases, ρ_m is matched with a simplex of a higher dimension. If $v = q_1 \frac{d}{2} + r$ then the path ends (to the left in (6.6)) at $\rho_m = \sigma'_{q_1+1, q_2}$. If $v = q_1 \frac{d}{2} + r + \frac{d}{2}$ then the path ends at σ_{q_1, q_2} . Finally, if $v = n - q_2 \frac{d}{2}$ then the path ends at σ'_{q_1, q_2+1} .

Part 4: incidence numbers. We have seven families of incidence numbers to compute, each coming from one of the alternating paths we have found.

- From σ'_{q_1+2, q_2} to σ_{q_1, q_2} . The alternating path is trivial and consists in removing the vertex $v = q_1 \frac{d}{2} + r$, so

$$[\sigma'_{q_1+2, q_2} : \sigma_{q_1, q_2}]^{\mathcal{M}} = (-1)^{|\{w \in \sigma_{q_1, q_2} | w < v\}|} = (-1)^{q_1 \frac{d}{2} + r}$$

which implies

$$[\sigma'_{q_1, q_2} : \sigma_{q_1-2, q_2}]^{\mathcal{M}} = (-1)^{q_1 \frac{d}{2} + r} = (-1)^\beta.$$

- From σ_{q_1, q_2+1} to σ_{q_1, q_2} . Again the path is trivial, and it consists in removing $v = n - q_2 \frac{d}{2}$. Therefore

$$\begin{aligned} [\sigma_{q_1, q_2+1} : \sigma_{q_1, q_2}]^{\mathcal{M}} &= (-1)^{|\{w \in \sigma_{q_1, q_2} | w < v\}|} \\ &= (-1)^{n - q_2 \frac{d}{2} - (k - q_1 - q_2 - 1)} \\ &= (-1)^{k \frac{d}{2} + r - q_2 \frac{d}{2} - k + q_1 + q_2 + 1} \\ &= (-1)^{(k - q_2) \left(\frac{d}{2} - 1\right) + r + q_1 + 1} \end{aligned}$$

which implies

$$\begin{aligned} [\sigma_{q_1, q_2} : \sigma_{q_1, q_2-1}]^{\mathcal{M}} &= (-1)^{(k - q_2 + 1) \left(\frac{d}{2} - 1\right) + r + q_1 + 1} \\ &= (-1)^\alpha. \end{aligned}$$

- From σ_{q_1+1, q_2} to σ_{q_1, q_2} . The path consists in removing $q_1 \frac{d}{2} + r$, adding $q_1 \frac{d}{2} + r + \frac{d}{2}$, and removing $q_1 \frac{d}{2} + r + d$. Therefore

$$\begin{aligned} [\sigma_{q_1+1, q_2} : \sigma_{q_1, q_2}]^{\mathcal{M}} &= (-1)(-1)^{q_1 \frac{d}{2} + r} (-1)^{q_1 \frac{d}{2} + r + \frac{d}{2} - 1} (-1)^{q_1 \frac{d}{2} + r + d - 1} \\ &= (-1)^{q_1 \frac{d}{2} + r + \frac{d}{2} + 1} \end{aligned}$$

which implies

$$\begin{aligned} [\sigma_{q_1, q_2} : \sigma_{q_1-1, q_2}]^{\mathcal{M}} &= (-1)^{(q_1-1) \frac{d}{2} + r + \frac{d}{2} + 1} \\ &= (-1)^{\beta+1}. \end{aligned}$$

- From σ'_{q_1, q_2+2} to σ_{q_1, q_2} . The path is the one we explicitly wrote at the end of Part 2. Notice that from ρ_{i-1} to τ_i one removes the vertex $v_i = n - (q_2 - i + 1) \frac{d}{2}$, and from τ_i to ρ_i one adds the vertex $v'_i = n - (q_2 - i - 1) \frac{d}{2}$. Therefore

$$[\rho_{i-1} : \tau_i][\rho_i : \tau_i] = (-1)^{|\{w \in \tau_i | v'_i < w < v_i\}|} = (-1)^{d-1} = -1.$$

Then we can compute the incidence number in the Morse complex:

$$[\sigma'_{q_1, q_2+2} : \sigma_{q_1, q_2}]^{\mathcal{M}} = (-1)^m \left(\prod_{i=1}^m [\rho_{i-1} : \tau_i][\rho_i : \tau_i] \right) [\sigma_{q_1, q_2} : \rho_m]$$

$$\begin{aligned} &= (-1)^m (-1)^m (-1)^{q_1 \frac{d}{2} + r + (d-1)} \\ &= (-1)^{q_1 \frac{d}{2} + r + 1} \end{aligned}$$

which implies

$$\begin{aligned} [\sigma'_{q_1, q_2} : \sigma_{q_1, q_2 - 2}]^{\mathcal{M}} &= (-1)^{q_1 \frac{d}{2} + r + 1} \\ &= (-1)^{\beta + 1}. \end{aligned}$$

- From σ'_{q_1+1, q_2} to σ'_{q_1, q_2} . The path is trivial and consists of removing $v = q_1 \frac{d}{2} + r$. Then

$$[\sigma'_{q_1+1, q_2} : \sigma'_{q_1, q_2}]^{\mathcal{M}} = (-1)^{q_1 \frac{d}{2} + r}$$

which implies

$$[\sigma'_{q_1, q_2} : \sigma'_{q_1-1, q_2}]^{\mathcal{M}} = (-1)^{(q_1-1) \frac{d}{2} + r} = (-1)^{\beta + \frac{d}{2}}.$$

- From σ_{q_1, q_2} to σ'_{q_1, q_2} . The path is trivial and consists in removing $v = q_1 \frac{d}{2} + r + \frac{d}{2}$. Then

$$[\sigma_{q_1, q_2} : \sigma'_{q_1, q_2}]^{\mathcal{M}} = (-1)^{q_1 \frac{d}{2} + r + \frac{d}{2} - 1} = (-1)^{\beta + \frac{d}{2} + 1}.$$

- From σ'_{q_1, q_2+1} to σ'_{q_1, q_2} . The path is trivial and consists in removing $v = n - q_2 \frac{d}{2}$. Then

$$\begin{aligned} [\sigma'_{q_1, q_2+1} : \sigma'_{q_1, q_2}]^{\mathcal{M}} &= (-1)^{n - q_2 \frac{d}{2} - (k - q_1 - q_2)} \\ &= (-1)^{k \frac{d}{2} + r - q_2 \frac{d}{2} - k + q_1 + q_2} \\ &= (-1)^{(k - q_2) \left(\frac{d}{2} - 1 \right) + r + q_1} \end{aligned}$$

which implies

$$\begin{aligned} [\sigma'_{q_1, q_2} : \sigma'_{q_1, q_2-1}]^{\mathcal{M}} &= (-1)^{(k - q_2 + 1) \left(\frac{d}{2} - 1 \right) + r + q_1} \\ &= (-1)^{\alpha + 1}. \end{aligned} \quad \square$$

We can finally compute the local homology $H_*(X_W; R)$ for Artin groups of type \tilde{C}_n . This result is new. The homology for $n \leq 6$ is also shown in Table 6.18 at the end of this chapter.

Theorem 6.12.5 (Homology in case \tilde{C}_n). Let G_W be an Artin group of type \tilde{C}_n . Then the φ_d -primary component of $H_*(X_W; R)$ is trivial for d odd, and for d even is as follows:

$$H_m(X_W; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus m+k-n+1} & \text{if } n-k \leq m \leq n-1 \\ 0 & \text{otherwise,} \end{cases}$$

where $n = k \frac{d}{2} + r$.

Proof. In order to apply Theorem 6.4.1 we need to find the rank of the boundary maps $\delta_{m+1}^{\varphi_d}$ of the Morse complex. For d odd there is only one critical cell, thus all the boundaries vanish. Suppose from now on that d is even, and let $n = k\frac{d}{2} + r$. In order to have a non-trivial boundary $\delta_{m+1}^{\varphi_d}$, we must have at least one critical simplex both in dimension m and in dimension $m + 1$, thus $m = n - k + l$ with $0 \leq l \leq k - 1$ (see Table 6.11).

Case 1: $l \leq k - 2$. We are going to prove that, for $l \leq k - 2$, a basis for the image of $\delta = \delta_{m+1}^{\varphi_d}$ is given by

$$\mathcal{B} = \{\delta\sigma_{q_1, q_2} \mid q_1 + q_2 = l\}.$$

By Theorem 6.12.4, we obtain the following formula for $\delta\sigma_{q_1, q_2}$:

$$\delta\sigma_{q_1, q_2} = (-1)^\alpha \sigma_{q_1, q_2-1} + (-1)^{\beta+1} \sigma_{q_1-1, q_2} + (-1)^{\beta+\frac{d}{2}+1} \sigma'_{q_1, q_2},$$

where $\alpha = (k - q_2)(\frac{d}{2} - 1) + q_1 + r + \frac{d}{2}$ and $\beta = q_1\frac{d}{2} + r$. In this formula the σ_{q_1, q_2-1} (resp. σ_{q_1-1, q_2}) term vanishes if $q_2 = 0$ (resp. $q_1 = 0$). The term σ'_{q_1, q_2} appears in $\delta\sigma_{q_1, q_2}$ but not in any other element of \mathcal{B} , thus \mathcal{B} is a linearly independent set. In addition, if $q_1 + q_2 = l + 1$, we have that

$$\begin{aligned} \delta\sigma'_{q_1, q_2} &= (-1)^{\beta+1} \sigma_{q_1, q_2-2} + (-1)^\beta \sigma_{q_1-2, q_2} + (-1)^{\alpha+1} \sigma'_{q_1, q_2-1} + (-1)^{\beta+\frac{d}{2}} \sigma'_{q_1-1, q_2} \\ &= (-1)^{\alpha+\beta+\frac{d}{2}} \left((-1)^{\alpha+\frac{d}{2}+1} \sigma_{q_1, q_2-2} + (-1)^{\beta+1} \sigma_{q_1-1, q_2-1} + (-1)^{\beta+\frac{d}{2}+1} \sigma'_{q_1, q_2-1} \right) \\ &\quad + (-1)^{\frac{d}{2}+1} \left((-1)^{\alpha+1} \sigma_{q_1-1, q_2-1} + (-1)^{\beta+\frac{d}{2}+1} \sigma_{q_1-2, q_2} + (-1)^{\beta+1} \sigma'_{q_1-1, q_2} \right) \\ &= (-1)^{\alpha+\beta+\frac{d}{2}} \delta\sigma_{q_1, q_2-1} + (-1)^{\frac{d}{2}+1} \delta\sigma_{q_1-1, q_2}. \end{aligned}$$

Therefore \mathcal{B} generates the image of $\delta_{m+1}^{\varphi_d}$. Thus $\text{rk } \delta_{m+1}^{\varphi_d} = |\mathcal{B}| = l + 1$ for $l \leq k - 2$.

Case 2: $l = k - 1$. For $l = k - 1$, i.e. $m = n - 1$, the situation is a bit different because there are no critical simplices of the form σ_{q_1, q_2} with $q_1 + q_2 = l$. However we can still define

$$\epsilon_{q_1, q_2} = (-1)^\alpha \sigma_{q_1, q_2-1} + (-1)^{\beta+1} \sigma_{q_1-1, q_2} + (-1)^{\beta+\frac{d}{2}+1} \sigma'_{q_1, q_2}$$

for $q_1 + q_2 = l$, and

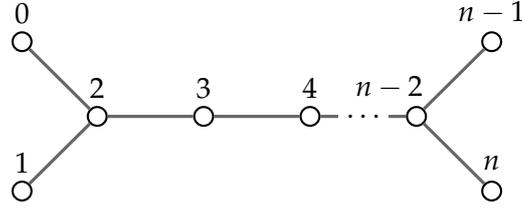
$$\mathcal{B}' = \{\epsilon_{q_1, q_2} \mid q_1 + q_2 = l\}.$$

The term σ'_{q_1, q_2} appears in ϵ_{q_1, q_2} but not in any other element of \mathcal{B}' , thus \mathcal{B}' is a linearly independent set. As above we have that, for $q_1 + q_2 = l + 1$,

$$\delta\sigma'_{q_1, q_2} = (-1)^{\alpha+\beta+\frac{d}{2}} \epsilon_{q_1, q_2-1} + (-1)^{\frac{d}{2}+1} \epsilon_{q_1-1, q_2}.$$

Then \mathcal{B}' generates (and so it is a basis of) the image of $\delta_n^{\varphi_d}$.

We have proved that, for $0 \leq l \leq k - 1$, the rank of $\delta_{m+1}^{\varphi_d}$ is equal to $l + 1 = m + k - n + 1$. Then we conclude by applying Theorem 6.4.1. \square


 Figure 6.11: Coxeter graph of type \tilde{D}_n .

6.13 CASE \tilde{D}_n

In this section we consider a Coxeter system (W, S) of type \tilde{D}_n , for $n \geq 4$ (see Figure 6.11). We are going to describe a φ_d -precise matching on $K = \mathcal{K}_W$. Again, this will be easy for d odd and quite involved for d even.

Matching 6.13.1 (φ_d -matching for d odd).

- (a) If $\sigma = \{1, 2, \dots, n\}$, then σ is critical.
- (b) If $1 \notin \sigma$, relabel vertices $\{n, n-1, \dots, 3, 2, 0\}$ as $\{1, 2, \dots, n\}$ and generate the matching as in K_n^D (see Matching 6.9.1).
- (c) In all remaining cases, match σ with $\sigma \vee 0$.

Matching 6.13.2 (φ_d -matching for d even). For $n = 4$ and $d \leq 6$, we construct the matching by hand, as follows.

- Case $n = 4, d = 2$. If $|\sigma| = 1$ and $2 \notin \sigma$, or $|\sigma| = 2$, or $|\sigma| = 3$ and $2 \in \sigma$, then match σ with $\sigma \vee 2$. Otherwise, σ is critical.
- Case $n = 4, d = 4$. If $2 \notin \sigma$ or $\sigma \cap \{1, 3, 4\} = \emptyset$, then match σ with $\sigma \vee 0$. Otherwise, σ is critical.
- Case $n = 4, d = 6$. Match σ with $\sigma \vee 0$, except in the following two cases: $2 \in \sigma, 0 \notin \sigma$ and $|\sigma| \geq 3$; or, $\{0, 2\} \subseteq \sigma$ and $|\sigma| = 4$.

In the remaining cases ($n \geq 5$ or $d \geq 8$), the matching is defined as follows.

- (a) If $1 \notin \sigma$, relabel vertices $\{n, n-1, \dots, 3, 2, 0\}$ as $\{1, 2, \dots, n\}$ and construct the matching as in K_n^D (see Matching 6.9.2).
- (b) Otherwise, if $d = 2$ and $\{0, 1, 2, 3\} \not\subseteq \sigma$, proceed as follows.
 - (b1) If $\{0, 1, 3\} \subseteq \sigma$, match σ with $\sigma \vee 4$ if $\{5, 6, \dots, n\} \not\subseteq \sigma$, else σ is critical.
 - (b2) Otherwise, if $\{1, 3, 4, \dots, n\} \not\subseteq \sigma$ then match σ with $\sigma \vee 2$, else σ is critical.
- (c) Otherwise, if $d \geq 4$ and $0 \notin \sigma$, proceed as follows.

- (c1) If $\{1, 2, \dots, \frac{d}{2}\} \subseteq \sigma$, relabel vertices $\{n, n-1, \dots, 2, 1\}$ as $\{1, 2, \dots, n\}$ and construct the matching as in $K_{n, \frac{d}{2}}^D$ (see Matching 6.9.2).
- (c2) Otherwise, if $n = \frac{d}{2} + 1$ and $\sigma = \{1, 2, \dots, n-2, n\}$, then σ is critical.
- (c3) Otherwise, if $\sigma = \{1, 2, \dots, n\}$, then σ is critical.
- (c4) Otherwise, match σ with $\sigma \cup \{0\}$.
- (d) Otherwise, if $d \geq 4$ and $2 \notin \sigma$, match σ with $\sigma \vee 0$.
- (e) Otherwise, if $d = 4$ and $3 \notin \sigma$, ignore vertices $0, 1, 2$, relabel vertices $\{n, n-1, \dots, 4\}$ as $\{1, 2, \dots, n-3\}$ and construct the matching as in K_{n-3}^D (see Matching 6.9.2).
- (f) Otherwise, if $d \geq 6$ and $3 \notin \sigma$, match σ with $\sigma \vee 0$.
- (g) Otherwise, proceed as follows. Recall that at this point $\{0, 1, 2, 3\} \subseteq \sigma$. Let $k \geq 4$ be the size of the leftmost connected component $\Gamma_0(\sigma)$ of the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ . Notice that $\{0, 1, \dots, k-1\} \subseteq \sigma$, unless $k = n$ and $\sigma = \{0, 1, \dots, n-2, n\}$. Similarly to Matching 6.9.2, write $k = q\frac{d}{2} + r$ where:

$$\begin{cases} 0 < r < \frac{d}{2} & \text{if } k \not\equiv 0 \pmod{\frac{d}{2}} \\ r \in \{0, \frac{d}{2}\} \text{ and } q \text{ even} & \text{if } k \equiv 0 \pmod{\frac{d}{2}}. \end{cases}$$

Define a vertex v as follows:

$$v = \begin{cases} q\frac{d}{2} & \text{if } q \text{ is even} \\ q\frac{d}{2} + 1 & \text{if } q \text{ is odd.} \end{cases}$$

- (g1) If $d = 4$, q odd and $r = 1$, proceed as follows.
- (g1.1) If $k \leq n-2$, ignore vertices $0, \dots, k$, relabel vertices $\{n, n-1, \dots, k+1\}$ as $\{1, 2, \dots, n-k\}$, and construct the matching as in K_{n-k}^D (see Matching 6.9.2).
- (g1.2) Otherwise, σ is critical.
- (g2) Otherwise, if $\sigma = \{0, 1, \dots, n-2, n\}$ and $v \geq n-1$, then σ is critical.
- (g3) Otherwise, if $v \in \sigma$, match σ with $\sigma \vee v$.
- (g4) Otherwise, if $v > n$, then σ is critical.
- (g5) Otherwise, proceed as follows. Let c be the size of the (possibly empty) connected component $C = \Gamma_{v+1}(\sigma)$ of the vertex $v+1$, in the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ . Let

$$\ell = \begin{cases} \frac{d}{2} & \text{if } q \text{ even} \\ \frac{d}{2} - 2 & \text{if } q \text{ odd.} \end{cases}$$

- (g5.1) If $\{n-1, n\} \subseteq C$, then σ is critical.

- (g5.2) Otherwise, if $c < \ell$, match σ with $\sigma \cup \{v\}$.
- (g5.3) Otherwise, if $c = \ell$, $n - 1 \notin C$ and $n \in C$, then σ is critical.
- (g5.4) Otherwise, ignore vertices $0, 1, \dots, v - 1$, relabel vertices $\{n, n - 1, \dots, v + 1\}$ as $\{1, 2, \dots, n - v\}$ and construct the matching as in $K_{n-v, \ell}^D$ (see Matching 6.9.2).

Lemma 6.13.3. Matchings 6.13.1 and 6.13.2 are φ_d -precise. In addition, the critical simplices of Matching 6.13.1 are those listed in Table 6.12.

Sketch of proof. We only discuss the critical simplices of Matching 6.13.1 (the case d odd), and see why this matching is precise. The check for Matching 6.13.2 is much more involved and is omitted.

Following the definition of Matching 6.13.1, one critical simplex is always given by $\bar{\sigma} = \{1, 2, \dots, n\}$. It has size $|\bar{\sigma}| = n$ and weight $v_{\varphi_d}(\bar{\sigma}) = \lfloor \frac{n}{d} \rfloor$. The other critical simplices arise from the matching on K_n^D , and therefore from the matching on K_{n-1}^A . There are two of them (which we denote by $\bar{\tau}_1$ and $\bar{\tau}_2$) when $n \equiv 0, 1 \pmod{d}$, and zero otherwise.

If $n \equiv 0 \pmod{d}$, the sizes of $\bar{\tau}_1$ and $\bar{\tau}_2$ are $n - 2\frac{n}{d} + 1$ and $n - 2\frac{n}{d}$, and their weights are 1 and 0, respectively. For $\frac{n}{d} \geq 2$ we have that $|\bar{\tau}_1| \leq |\bar{\sigma}| - 2$, so the incidence number between $\bar{\sigma}$ and any of $\bar{\tau}_1$ and $\bar{\tau}_2$ is zero. This is enough to conclude that the matching is φ_d -precise, since $|\tau_1| - v_{\varphi_d}(\tau_1) = |\tau_2| - v_{\varphi_d}(\tau_2)$. The remaining case is $\frac{n}{d} = 1$, i.e. $n = d$. Here we have $\bar{\tau}_1 = \{0, 2, 3, \dots, n - 2, n - 1\}$ and $\bar{\tau}_2 = \{0, 2, 3, \dots, n - 2\}$. There are exactly two alternating paths from $\bar{\sigma}$ to $\bar{\tau}_1$:

- $\{1, 2, \dots, n\} \succ \{1, 2, \dots, n - 1\} \prec \{0, 1, \dots, n - 1\} \succ \{0, 2, 3, \dots, n - 1\}$;
- $\{1, 2, \dots, n\} \succ \{1, 2, \dots, n - 2, n\} \prec \{0, 1, \dots, n - 2, n\} \succ \{0, 2, 3, \dots, n - 2, n\} \prec \{0, 2, 3, \dots, n\} \succ \{0, 2, 3, \dots, n - 1\}$.

These two paths give opposite contributions to the incidence number $[\bar{\sigma} : \bar{\tau}_1]^{\mathcal{M}}$, so $[\bar{\sigma} : \bar{\tau}_1]^{\mathcal{M}} = 0$. Therefore the matching is φ_d -precise.

If $n \equiv 1 \pmod{d}$, the sizes of $\bar{\tau}_1$ and $\bar{\tau}_2$ are $n - 2\frac{n-1}{d}$ and $n - 2\frac{n-1}{d} - 1$, and their weights are 1 and 0, respectively. Then $|\bar{\tau}_1| \leq |\bar{\sigma}| - 2$, so the incidence number between $\bar{\sigma}$ and any of $\bar{\tau}_1$ and $\bar{\tau}_2$ is always zero. \square

6.14 EXCEPTIONAL CASES

Consider first the case $I_2(m)$ for $m \geq 5$ (see Figure 6.12). The Poincaré polynomial is given by $W_{I_2(m)} = [2]_q [m]_q$. Then the φ_d -weight is

$$\omega_{\varphi_d}(I_2(m)) = \begin{cases} 2 & \text{if } d = 2 \text{ and } m \text{ even} \\ 1 & \text{if } d = 2 \text{ and } m \text{ odd} \\ 1 & \text{if } d \geq 3 \text{ and } d \mid m \\ 0 & \text{if } d \geq 3 \text{ and } d \nmid m. \end{cases}$$

Case	# Critical	$ \sigma - v_{\varphi_d}(\sigma)$
$n \equiv 0 \pmod{d}$	3	$n - \frac{n}{d}$ (once), $n - 2\frac{n}{d}$ (twice)
$n \equiv 1 \pmod{d}$		$n - \frac{n-1}{d}$ (once), $n - 2\frac{n-1}{d} - 1$ (twice)
else	1	$n - \lfloor \frac{n}{d} \rfloor$

Table 6.12: Critical simplices of Matching 6.13.1 (case \tilde{D}_n , d odd).

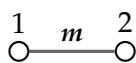


Figure 6.12: Coxeter graph of type $I_2(m)$.

In this case φ_d -precise matchings are easy to construct by hand. As a straightforward consequence we also obtain the homology groups $H_*(X_W; R)$.

Matching 6.14.1 (φ_d -matching on $\mathcal{K}_{W_{I_2(m)}}$).

- If $d = 2$ and m is even, every simplex is critical. Critical simplices are then: $\{1, 2\}$ (size 2, weight 2), $\{1\}$, $\{2\}$ (size 1, weight 1), and \emptyset (size 0, weight 0). By Theorem 6.4.1, the homology groups are: $H_0(X_W; R)_{\varphi_2} \cong \frac{R}{(\varphi_2)}$, and $H_1(X_W; R)_{\varphi_2} \cong \frac{R}{(\varphi_2)}$.
- If $d = 2$ and m is odd, match $\{1, 2\}$ with $\{1\}$ (both simplices have weight 1). The critical simplices are: $\{2\}$ (size 1, weight 1), and \emptyset (size 0, weight 0). The homology groups are: $H_0(X_W; R)_{\varphi_2} \cong \frac{R}{(\varphi_2)}$, and $H_1(X_W; R)_{\varphi_2} \cong 0$.
- If $d \geq 3$ and $d \mid m$, match $\{2\}$ with \emptyset (both simplices have weight 0). The critical simplices are: $\{1, 2\}$ (size 2, weight 1), and $\{1\}$ (size 1, weight 0). The homology groups are: $H_0(X_W; R)_{\varphi_d} \cong 0$, and $H_1(X_W; R)_{\varphi_d} \cong \frac{R}{(\varphi_d)}$.
- If $d \geq 3$ and $d \nmid m$, match $\{1, 2\}$ with $\{1\}$ and $\{2\}$ with \emptyset (all simplices have weight 0). There are no critical simplices, and all homology groups are trivial.

Theorem 6.14.2 (Homology in case $I_2(m)$). Let G_W be an Artin group of type $I_2(m)$. Then its local homology is given by

$$H_0(X_W; R) \cong \frac{R}{(\varphi_2)}, \quad H_1(X_W; R) \cong \bigoplus_{\substack{d \mid m \\ d \geq 2}} \frac{R}{(\varphi_d)}. \quad \square$$

This result corrects the one given in [DCPSS99], where proper divisors of m were not taken into account.

We constructed precise matchings by means of a computer program for all other exceptional spherical and affine cases (see Figures 2.6 and 2.8). The explicit description of these matchings, together with proofs of preciseness and homology computations, can be obtained through the software library [Pao17b].

The homology groups can be computed using Theorem 6.4.1, and are described in Tables 6.20 and 6.21. We recover the results of [DCPSS99] (for the spherical cases) and [SV13] (for the affine cases), except for minor corrections in the cases E_8 and \tilde{E}_8 .

6.15 LOCAL HOMOLOGY OF LOW-DIMENSIONAL GROUPS

In the last part of this chapter, we provide tables with the local homology of Artin groups of low dimension. This includes the homology of all exceptional spherical and affine groups. Computations were carried out with the software library [Pao17b].

Our results mostly agree with previous computations [Fre88, DCPSS99, DCPS01, CMS08a, CMS08b, CMS10], as discussed throughout this chapter. In the tables we employ the notation $\{d\} = R/(q_d)$, as in [DCPSS99, DCPS01].

6.15. Local homology of low-dimensional groups

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
H_0	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}
H_1		{3}	{3}	0	0	0	0	0	0	0
H_2			{4}	{4}	{3}	{3}	0	0	0	0
H_3				{5}	{5}	0	0	{3}	{3}	0
H_4					{6}	{6}	{4}	{4}	0	0
H_5						{7}	{7}	0	0	0
H_6							{8}	{8}	{5}	{5}
H_7								{9}	{9}	0
H_8									{10}	{10}
H_9										{11}

Table 6.13: Homology in case A_n for $n \leq 10$.

	B_2	B_3	B_4	B_5	B_6	B_7	B_8
H_0	{2}	{2}	{2}	{2}	{2}	{2}	{2}
H_1	Δ_{B_2}	{2}	{2}	{2}	{2}	{2}	{2}
H_2		Δ_{B_3}	{2} \oplus {4}	{2}	{2}	{2}	{2}
H_3			Δ_{B_4}	{2}	{2} \oplus {4}	{2}	{2}
H_4				Δ_{B_5}	{2} \oplus {4} \oplus {6}	{2}	{2} \oplus {4}
H_5					Δ_{B_6}	{2}	{2} \oplus {4}
H_6						Δ_{B_7}	{2} \oplus {4} \oplus {8}
H_7							Δ_{B_8}

$$\begin{aligned} \Delta_{B_2} &= \{2\} \oplus \{4\} \\ \Delta_{B_3} &= \{2\} \oplus \{6\} \\ \Delta_{B_4} &= \{2\} \oplus \{4\} \oplus \{8\} \\ \Delta_{B_5} &= \{2\} \oplus \{10\} \\ \Delta_{B_6} &= \{2\} \oplus \{4\} \oplus \{6\} \oplus \{12\} \\ \Delta_{B_7} &= \{2\} \oplus \{14\} \\ \Delta_{B_8} &= \{2\} \oplus \{4\} \oplus \{8\} \oplus \{16\} \end{aligned}$$

Table 6.14: Homology in case B_n for $n \leq 8$.

6. LOCAL HOMOLOGY OF ARTIN GROUPS

	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	$\{3\}$	0	0	0	0	0	0
H_2	$\{4\}^2$	$\{4\}$	$\{3\}$	$\{3\}$	0	0	0
H_3	Δ_{D_4}	$\{5\}$	$\{5\}$	0	0	$\{3\}$	$\{3\}$
H_4		Δ_{D_5}	$\{6\}^2$	$\{4\} \oplus \{6\}$	$\{4\}^2$	$\{4\}$	0
H_5			Δ_{D_6}	$\{4\} \oplus \{7\}$	$\{4\} \oplus \{7\}$	0	0
H_6				Δ_{D_7}	$\{4\} \oplus \{8\}^2$	$\{8\}$	$\{5\}$
H_7					Δ_{D_8}	$\{9\}$	$\{6\} \oplus \{9\}$
H_8						Δ_{D_9}	$\{6\} \oplus \{10\}^2$
H_9							$\Delta_{D_{10}}$

$$\begin{aligned} \Delta_{D_4} &= \{2\} \oplus \{4\} \oplus \{6\} \\ \Delta_{D_5} &= \{8\} \\ \Delta_{D_6} &= \{2\} \oplus \{6\} \oplus \{10\} \\ \Delta_{D_7} &= \{4\} \oplus \{12\} \\ \Delta_{D_8} &= \{2\} \oplus \{4\} \oplus \{8\} \oplus \{14\} \\ \Delta_{D_9} &= \{16\} \\ \Delta_{D_{10}} &= \{2\} \oplus \{6\} \oplus \{10\} \oplus \{18\} \end{aligned}$$

Table 6.15: Homology in case D_n for $n \leq 10$.

	\tilde{A}_2	\tilde{A}_3	\tilde{A}_4	\tilde{A}_5	\tilde{A}_6
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	$\{3\}^2$	$\{3\}$	0	0	0
H_2	R	$\{2\} \oplus \{4\}^3$	$\{2\} \oplus \{3\} \oplus \{4\}$	$\{2\} \oplus \{3\}^2$	$\{2\} \oplus \{3\}$
H_3		R	$\{5\}^4$	$\{4\} \oplus \{5\}$	0
H_4			R	$\{2\} \oplus \{3\}^2 \oplus \{6\}^5$	$\Delta_{\tilde{A}_6}$
H_5				R	$\{7\}^6$
H_6					R

$$\Delta_{\tilde{A}_6} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\}$$

Table 6.16: Homology in case \tilde{A}_n for $n \leq 6$.

6.15. Local homology of low-dimensional groups

	\tilde{B}_3	\tilde{B}_4	\tilde{B}_5	\tilde{B}_6	\tilde{B}_7
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_2	$\Delta_{\tilde{B}_3}$	$\{2\} \oplus \{4\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_3	R	$\Delta_{\tilde{B}_4}$	$\{2\} \oplus \{4\}$	$\{2\} \oplus \{4\}$	$\{2\}$
H_4		R	$\Delta_{\tilde{B}_5}$	$\{2\} \oplus \{4\} \oplus \{6\}$	$\{2\} \oplus \{4\}$
H_5			R	$\Delta_{\tilde{B}_6}$	$\{2\} \oplus \{4\}^2 \oplus \{6\}$
H_6				R	$\Delta_{\tilde{B}_7}$
H_7					R

$$\begin{aligned}\Delta_{\tilde{B}_3} &= \{2\}^2 \oplus \{4\} \oplus \{6\} \\ \Delta_{\tilde{B}_4} &= \{2\}^3 \oplus \{4\}^2 \oplus \{6\} \oplus \{8\} \\ \Delta_{\tilde{B}_5} &= \{2\}^3 \oplus \{4\}^2 \oplus \{6\} \oplus \{8\} \oplus \{10\} \\ \Delta_{\tilde{B}_6} &= \{2\}^4 \oplus \{4\}^2 \oplus \{6\}^2 \oplus \{8\} \oplus \{10\} \oplus \{12\} \\ \Delta_{\tilde{B}_7} &= \{2\}^4 \oplus \{4\}^3 \oplus \{6\}^2 \oplus \{8\} \oplus \{10\} \oplus \{12\} \oplus \{14\}\end{aligned}$$

Table 6.17: Homology in case \tilde{B}_n for $n \leq 7$.

	\tilde{C}_2	\tilde{C}_3	\tilde{C}_4	\tilde{C}_5	\tilde{C}_6
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	$\Delta_{\tilde{C}_2}$	$\{2\}^2$	$\{2\}^2$	$\{2\}^2$	$\{2\}^2$
H_2	R	$\Delta_{\tilde{C}_3}$	$\{2\}^3 \oplus \{4\}$	$\{2\}^3$	$\{2\}^3$
H_3		R	$\Delta_{\tilde{C}_4}$	$\{2\}^4 \oplus \{4\}$	$\{2\}^4 \oplus \{4\}$
H_4			R	$\Delta_{\tilde{C}_5}$	$\{2\}^5 \oplus \{4\}^2 \oplus \{6\}$
H_5				R	$\Delta_{\tilde{C}_6}$
H_6					R

$$\begin{aligned}\Delta_{\tilde{C}_2} &= \{2\}^2 \oplus \{4\} \\ \Delta_{\tilde{C}_3} &= \{2\}^3 \oplus \{4\} \oplus \{6\} \\ \Delta_{\tilde{C}_4} &= \{2\}^4 \oplus \{4\}^2 \oplus \{6\} \oplus \{8\} \\ \Delta_{\tilde{C}_5} &= \{2\}^5 \oplus \{4\}^2 \oplus \{6\} \oplus \{8\} \oplus \{10\} \\ \Delta_{\tilde{C}_6} &= \{2\}^6 \oplus \{4\}^3 \oplus \{6\}^2 \oplus \{8\} \oplus \{10\} \oplus \{12\}\end{aligned}$$

Table 6.18: Homology in case \tilde{C}_n for $n \leq 6$.

6. LOCAL HOMOLOGY OF ARTIN GROUPS

	\tilde{D}_4	\tilde{D}_5	\tilde{D}_6	\tilde{D}_7	\tilde{D}_8	\tilde{D}_9
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	$\{3\}$	0	0	0	0	0
H_2	$\{4\}^3$	$\{4\}$	$\{3\}$	$\{3\}$	0	0
H_3	$\Delta_{\tilde{D}_4}$	$\{5\}$	$\{5\}$	0	0	$\{3\}$
H_4	R	$\Delta_{\tilde{D}_5}$	$\{4\} \oplus \{6\}^3$	$\{4\}^3 \oplus \{6\}$	$\{4\}^3$	$\{4\}$
H_5		R	$\Delta_{\tilde{D}_6}$	$\{4\}^4 \oplus \{7\}$	$\{4\}^2 \oplus \{7\}$	0
H_6			R	$\Delta_{\tilde{D}_7}$	$\{4\}^2 \oplus \{6\} \oplus \{8\}^3$	$\{8\}$
H_7				R	$\Delta_{\tilde{D}_8}$	$\{6\} \oplus \{9\}$
H_8					R	$\Delta_{\tilde{D}_9}$
H_9						R

$$\Delta_{\tilde{D}_4} = \{2\}^4 \oplus \{4\}^3 \oplus \{6\}^3$$

$$\Delta_{\tilde{D}_5} = \{2\}^2 \oplus \{4\} \oplus \{6\} \oplus \{8\}^3$$

$$\Delta_{\tilde{D}_6} = \{2\}^5 \oplus \{4\}^2 \oplus \{6\}^3 \oplus \{8\} \oplus \{10\}^3$$

$$\Delta_{\tilde{D}_7} = \{2\}^3 \oplus \{4\}^5 \oplus \{6\} \oplus \{8\} \oplus \{10\} \oplus \{12\}^3$$

$$\Delta_{\tilde{D}_8} = \{2\}^6 \oplus \{4\}^4 \oplus \{6\}^2 \oplus \{8\}^3 \oplus \{10\} \oplus \{12\} \oplus \{14\}^3$$

$$\Delta_{\tilde{D}_9} = \{2\}^4 \oplus \{4\}^2 \oplus \{6\}^2 \oplus \{8\} \oplus \{10\} \oplus \{12\} \oplus \{14\} \oplus \{16\}^3$$

Table 6.19: Homology in case \tilde{D}_n for $n \leq 9$.

	$I_2(m)$	H_3	H_4	F_4	E_6	E_7	E_8
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	$\Delta_{I_2(m)}$	0	0	$\{2\}$	0	0	0
H_2		Δ_{H_3}	0	$\{2\} \oplus \{3\} \oplus \{6\}$	0	0	0
H_3			Δ_{H_4}	Δ_{F_4}	0	0	0
H_4					$\{6\} \oplus \{8\}$	$\{6\}$	$\{4\}$
H_5					Δ_{E_6}	$\{6\}$	0
H_6						Δ_{E_7}	$\{8\} \oplus \{12\}$
H_7							Δ_{E_8}

$$\Delta_{I_2(m)} = \bigoplus_{\substack{d|m \\ d \geq 2}} \{d\}$$

$$\Delta_{H_3} = \{2\} \oplus \{6\} \oplus \{10\}$$

$$\Delta_{H_4} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\} \oplus \{10\} \oplus \{12\} \oplus \{15\} \oplus \{20\} \oplus \{30\}$$

$$\Delta_{F_4} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{6\} \oplus \{8\} \oplus \{12\}$$

$$\Delta_{E_6} = \{3\} \oplus \{6\} \oplus \{9\} \oplus \{12\}$$

$$\Delta_{E_7} = \{2\} \oplus \{6\} \oplus \{14\} \oplus \{18\}$$

$$\Delta_{E_8} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\} \oplus \{8\} \oplus \{10\} \oplus \{12\} \oplus \{15\} \oplus \{20\} \\ \oplus \{24\} \oplus \{30\}$$

Table 6.20: Homology in the exceptional spherical cases.

6. LOCAL HOMOLOGY OF ARTIN GROUPS

	\tilde{I}_1	\tilde{G}_2	\tilde{F}_4	\tilde{E}_6	\tilde{E}_7	\tilde{E}_8
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	R	$\{2\} \oplus \{3\}$	$\{2\}$	0	0	0
H_2		R	$\{2\} \oplus \{3\}$	0	0	0
H_3			$\Delta_{\tilde{F}_4}$	$\{3\}$	$\{3\}$	0
H_4			R	$\{5\} \oplus \{8\}$	0	$\{4\}$
H_5				$\Delta_{\tilde{E}_6}$	0	0
H_6				R	$\Delta_{\tilde{E}_7}$	$\{5\} \oplus \{8\}$
H_7					R	$\Delta_{\tilde{E}_8}$
H_8						R

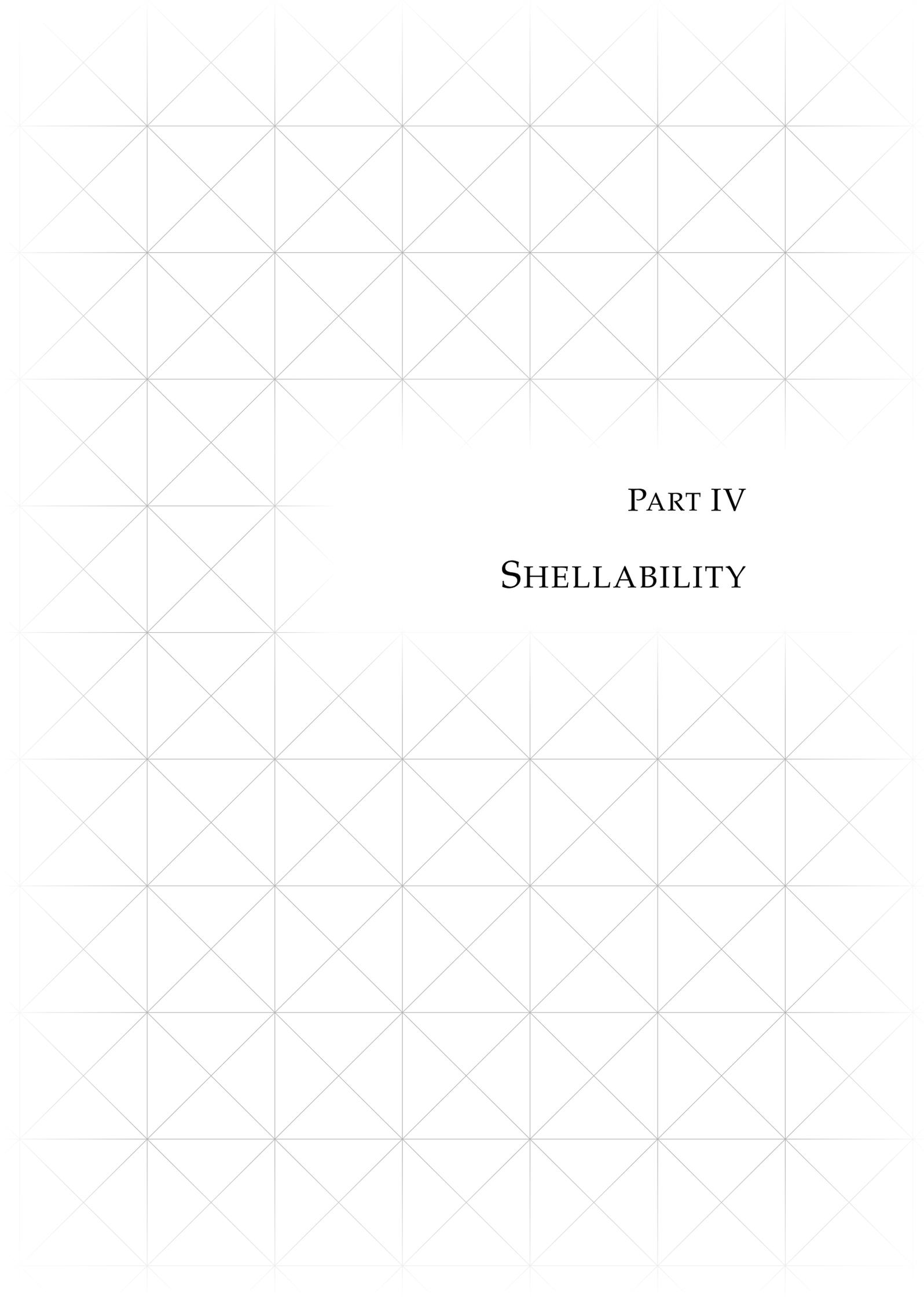
$$\Delta_{\tilde{F}_4} = \{2\}^2 \oplus \{3\} \oplus \{4\} \oplus \{8\}$$

$$\Delta_{\tilde{E}_6} = \{2\} \oplus \{3\}^3 \oplus \{6\}^2 \oplus \{9\}^2 \oplus \{12\}^2$$

$$\Delta_{\tilde{E}_7} = \{2\}^3 \oplus \{3\} \oplus \{4\} \oplus \{6\} \oplus \{8\} \oplus \{10\} \oplus \{14\} \oplus \{18\}$$

$$\Delta_{\tilde{E}_8} = \{2\}^2 \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{8\} \oplus \{9\} \oplus \{14\}$$

Table 6.21: Homology in the exceptional affine cases.



PART IV

SHELLABILITY

Shellability of arrangements defined by root systems

7.1 INTRODUCTION

The material of this chapter is based on a joint work with Emanuele Delucchi and Noriane Girard [DGP17].

Posets of flats of linear hyperplane arrangements (and, more generally, geometric lattices) are shellable. Indeed, geometric lattices can be characterized by their particularly nice shellability properties [DH14], and the number of spheres in the homotopy type of their order complex is an evaluation of the characteristic polynomial of the associated matroid. It is therefore natural to ask whether posets of layers of toric and elliptic arrangements are shellable (see Section 2.7).

A general approach to this question is made difficult by the fact that, in contrast with the case of hyperplane arrangements and geometric lattices, little is known about the abstract properties of posets of layers of toric and elliptic arrangements. In [DR18], these posets are studied as quotients of geometric semilattices by translative group actions. Then one could ask if such quotients of geometric semilattices are shellable in general. Unfortunately this is not the case, since they can have torsion in homology: this happens e.g. in the case of the antipodal action of \mathbb{Z}_2 on the six hyperplanes bounding a cube.

In this chapter we focus on the case of arrangements associated to root systems, where we can take advantage of Bibby's description of the posets of layers as certain posets of labeled partitions [Bib18]. This leads us to introduce a two-parameter class of posets of labeled partitions which we prove to be EL-shellable. The posets of layers of linear, toric or elliptic arrangements of type A , B , C , and D are subsets of elements of our two-parameter class, and the induced labelings are EL-labelings. In particular, they are homotopy

equivalent to a wedge of spheres. We summarize the results of this chapter in the following theorem.

Theorem. Let \mathcal{A} be a (linear, toric or elliptic) arrangement defined by a root system. Then the poset of layers $\mathcal{L}(\mathcal{A})$ is EL-shellable. In particular, the order complex of the poset $\tilde{\mathcal{L}}(\mathcal{A})$ (obtained by removing the minimum $\hat{0}$ and the maximum $\hat{1}$ if necessary) is homotopy equivalent to a wedge of spheres of dimension equal to the rank of $\tilde{\mathcal{L}}(\mathcal{A})$. Closed formulas for the number of spheres are given in Table 7.1.

Type	Linear	Toric	Elliptic
A_n	$n!$	$n!$	$n!$
B_n	$(2n - 1)!!$	$(2n - 3)!!(n - 1)$	[...]
C_n		$(2n - 1)!!$	$(2n + 1)!!$
D_n	$(2n - 3)!!(n - 1)$	$(2n - 5)!!(n^2 - 3n + 3)$	[...]

Table 7.1: Closed formulas for the number of spheres. In the two missing cases, the number of spheres can be obtained by setting $m = 4$ in the formulas of Theorem 7.6.2.

This chapter is structured as follows. In Section 7.2 we introduce the poset $\Pi_{n,\Sigma}$ of Σ -labeled partitions of the set $\{1, \bar{1}, \dots, n, \bar{n}\}$, where Σ is any finite set. In Section 7.3 we describe the four classes of subposets of $\Pi_{n,\Sigma}$ which, following Bibby [Bib18], are isomorphic to the posets of layers of linear ($|\Sigma| = 1$), toric ($|\Sigma| = 2$), or elliptic ($|\Sigma| = 4$) arrangements defined by the four infinite families of root systems. In Section 7.4 we prove that the poset $\Pi_{n,\Sigma}$ is shellable, by constructing an EL-labeling. Then, in Section 7.5 we prove that this labeling induces EL-labelings on all the above-mentioned subposets, corresponding to posets of layers of arrangements defined by root systems. In particular, the (reduced) order complexes are homotopy equivalent of a wedge of spheres. The homotopy type is hence determined by the number of these spheres. In Section 7.6 we tackle this enumeration problem: we give general formulas (for any Σ) and deduce closed expressions for most of the cases (see Table 7.1).

7.2 LABELED PARTITIONS

A *partition* of a finite set S is a collection $\pi = \{B_1, \dots, B_l\}$ of disjoint sets, called “blocks”, such that $\bigcup_i B_i = S$. We employ the notation

$$\pi = B_1 \mid \cdots \mid B_l.$$

Let Σ be a finite set (of “signs”). A *partition of S labeled by Σ* is a partition π of S together with a subset $T \subseteq \pi$ and an injection $f : T \rightarrow \Sigma$. The blocks in T are called *signed blocks* of π and those of $\pi \setminus T$ are called *unsigned blocks* of π .

For any given integer $n \in \mathbb{N}$, let

$$[n] = \{1, \dots, n\}, \quad [[n]] = \{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}\}.$$

Notice that the set $[[n]]$ carries a natural involution $\bar{\cdot} : [[n]] \rightarrow [[n]]$ defined as: $i \mapsto \bar{i}$, and $\bar{i} \mapsto i$, for all $i = 1, \dots, n$. In general, given a subset $A \subseteq [[n]]$, let

$$\bar{A} = \{\bar{a} \mid a \in A\}.$$

The finest partition of $[[n]]$ is the one in which every block is a singleton. We denote it by

$$\hat{0} = 1 \mid \bar{1} \mid 2 \mid \bar{2} \mid \cdots \mid n \mid \bar{n}.$$

Definition 7.2.1. Let Σ be a set (of “signs”). We say that π is a Σ -labeled partition of $[[n]]$ if π is a partition of $[[n]]$ labeled by Σ such that:

- for every $R \in \pi$, we also have $\bar{R} \in \pi$;
- $S = \bar{S}$ if and only if $S \in \pi$ is signed.

We will write this as follows:

$$\pi = S_{\sigma_1} \mid \cdots \mid S_{\sigma_l} \mid R_1 \mid \bar{R}_1 \mid \cdots \mid R_k \mid \bar{R}_k. \quad (7.1)$$

Here the signed blocks are denoted by S , and carry the sign $\sigma_i \in \Sigma$ as an index. The unsigned blocks are denoted by R_i . We call $\Sigma(\pi) = \{\sigma_1, \dots, \sigma_l\}$ the *sign* of π .

Given a Σ -labeled partition as in Equation (7.1) and a sign $\sigma \in \Sigma(\pi)$, we write $\pi_\sigma = S_\sigma$.

Definition 7.2.2. Let π be a Σ -labeled partition of $[[n]]$ written as in Equation (7.1), and let

$$\psi = S'_{\tau_1} \mid \cdots \mid S'_{\tau_m} \mid R'_1 \mid \bar{R}'_1 \mid \cdots \mid R'_q \mid \bar{R}'_q.$$

We say that π is a *refinement* of ψ if:

- $\Sigma(\pi) \subseteq \Sigma(\psi)$, and $\pi_\sigma \subseteq \psi_\sigma$ for every $\sigma \in \Sigma(\pi)$;
- for every $i \in [k]$, there exist two blocks of ψ , say T and \bar{T} , such that $R_i \subseteq T$ and $\bar{R}_i \subseteq \bar{T}$ (notice that T and \bar{T} can be equal if $T = \bar{T} = \psi_{\tau_j}$ for some $j \in [m]$).

We write $\pi \preceq \psi$ if π is a refinement of ψ . This defines a partial order relation on the set of all Σ -labeled partitions of $[[n]]$.

Definition 7.2.3. Let $\Pi_{n,\Sigma}$ be the poset of all Σ -labeled partitions of $[[n]]$, with a new element $\hat{1}$ added to the top, partially ordered by \preceq . Therefore, the element $\hat{1}$ covers all the maximal Σ -labeled partitions.

Remark 7.2.4. The poset $\Pi_{n,\Sigma}$ is ranked, with rank function given by: $\text{rk}(\pi) = n - k$ for $\pi \neq \hat{1}$ (where k is as in Equation (7.1)), and $\text{rk}(\hat{1}) = n + 1$.

7.3 POSETS OF LAYERS

The following is an explicit description of the posets of layers of (linear, toric, or elliptic) arrangements defined by root systems of type A_{n-1} , B_n , C_n , and D_n .

Theorem 7.3.1 (Barcelo-Ihrig [BI99, Theorem 4.1], Bibby [Bib18, Theorem 3.3]). The posets of layers of arrangements defined by root systems have the following form, where $|\Sigma| = 1$ in the linear case, $|\Sigma| = 2$ in the toric case, and $|\Sigma| = 4$ in the elliptic case.

Type A_{n-1} . $\mathcal{L}(\mathcal{A})$ is isomorphic to the subposet of $\Pi_{n,\Sigma} \setminus \{\hat{1}\}$ consisting of all elements of the form $R_1 \mid \bar{R}_1 \mid \cdots \mid R_k \mid \bar{R}_k$, where $R_1 \mid \cdots \mid R_k$ is a partition of $[n]$. Thus, in all cases $\mathcal{L}(\mathcal{A})$ is isomorphic to the classical lattice of all partitions of $[n]$ ordered by refinement.

Type B_n . Fix some distinguished element $\bar{\sigma} \in \Sigma$. Then $\mathcal{L}(\mathcal{A})$ is isomorphic to the subposet of $\Pi_{n,\Sigma} \setminus \{\hat{1}\}$ consisting of all elements $x \in \Pi_{n,\Sigma} \setminus \{\hat{1}\}$ such that $|x_\sigma| \neq 2$ whenever $\sigma \neq \bar{\sigma}$.

Type C_n . $\mathcal{L}(\mathcal{A})$ is isomorphic to $\Pi_{n,\Sigma} \setminus \{\hat{1}\}$.

Type D_n . $\mathcal{L}(\mathcal{A})$ is isomorphic to the subposet of $\Pi_{n,\Sigma} \setminus \{\hat{1}\}$ consisting of all elements $x \in \Pi_{n,\Sigma} \setminus \{\hat{1}\}$ such that $|x_\sigma| \neq 2$ for all $\sigma \in \Sigma$.

Our results will apply to the following more general families of posets.

Definition 7.3.2. For any $n \in \mathbb{N}$ and any finite set of signs Σ with a distinguished element $\bar{\sigma} \in \Sigma$, the poset of Σ -signed partitions of type A_{n-1} , B_n , C_n or D_n is the subposet of $\Pi_{n,\Sigma}$ satisfying the corresponding condition in Theorem 7.3.1.

Remark 7.3.3. All the subposets in Theorem 7.3.1 and Definition 7.3.2 are ranked, with rank function induced by $\Pi_{n,\Sigma}$.

Remark 7.3.4. When \mathcal{A} is an arrangement of hyperplanes, the characteristic polynomial of $\mathcal{L}(\mathcal{A})$ is known [OT13, Definition 2.52, Theorem 4.137, and Corollary 6.62]). In the toric case, Ardila, Castillo, and Henley [ACH14] gave formulas for the arithmetic Tutte polynomials of the associated arrangements, from which in principle the characteristic polynomial can be computed. In the elliptic case no explicit formula is known to us.

7.4 EL-SHELLABILITY OF POSETS OF LABELED PARTITIONS

For a block R of some Σ -labeled partition, define the *representative* of R as the minimum element $i \in [n]$ such that $i \in R$ or $\bar{i} \in R$. Denote by $r(R)$ the representative of R . Notice that R and \bar{R} share the same representative, and that the representative of R does not necessarily belong to R .

We call a block $R \subseteq [[n]]$ *normalized* if the representative of R belongs to R . For instance, $\{2, \bar{4}, 5\}$ is normalized and $\{\bar{2}, 4, 5\}$ is not. Notice that exactly one of R and \bar{R} is normalized, whenever R is an unsigned block.

An edge (x, y) of the Hasse diagram of $\Pi_{n, \Sigma}$ with $y \neq \hat{1}$ is called:

- of sign σ if $x_\sigma \subsetneq y_\sigma$ for a $\sigma \in \Sigma$;
- *coherent* if $R \cup R' \in y$ for some normalized unsigned blocks $R, R' \in x$ with $R \neq R'$;
- *non-coherent* if $R \cup \bar{R}' \in y$ for some normalized unsigned blocks $R, R' \in x$ with $R \neq R'$.

If $y \neq \hat{1}$, the edge (x, y) is of exactly one of these three types. We also say that (x, y) is *unsigned* if it is either coherent or non-coherent, and that it is *signed* otherwise.

Definition 7.4.1 (Edge labeling of $\Pi_{n, \Sigma}$). Given a total order $<$ of Σ , define the following edge labeling λ of $\Pi_{n, \Sigma}$.

- Let (x, y) be an unsigned edge. Let $R, R' \in x$ such that $R \cup R' \in y$, as above, and let i and j be the representatives of R and R' . Then

$$\lambda(x, y) = \begin{cases} (0, \max(i, j)) & \text{if } (x, y) \text{ is coherent} \\ (2, \min(i, j)) & \text{if } (x, y) \text{ is non-coherent.} \end{cases}$$

- Let (x, y) be an edge of sign σ . Then

$$\lambda(x, y) = \begin{cases} (1, |\Sigma(x)_{\leq \sigma}|) & \text{if } \sigma \in \Sigma(x) \\ (1, |\Sigma_{\leq \sigma} \cup \Sigma(x)|) & \text{otherwise.} \end{cases}$$

- For $x \prec \hat{1}$, let

$$\lambda(x, \hat{1}) = (1, 2).$$

Labels are ordered lexicographically.

Theorem 7.4.2 (EL-shellability of $\Pi_{n, \Sigma}$). The labeling λ of Definition 7.4.1 is an EL-labeling of $\Pi_{n, \Sigma}$.

Proof. The proof is divided into five parts.

Part 1: intervals $[x, z]$ that contain only coherent edges. Here x and z must have the same signed part. Thus, every such interval is isomorphic to an interval of the lattice of standard partitions of $[n] \setminus \bigcup_{\sigma \in \Sigma} x_\sigma$. The isomorphism maps an element $x_{\sigma_1} | \cdots | x_{\sigma_l} | R_1 | \bar{R}_1 | \cdots | R_k | \bar{R}_k \in [x, z]$ to $\tilde{R}_1 | \cdots | \tilde{R}_k$ where, for all i , we define $\tilde{R}_i = (R_i \cup \bar{R}_i) \cap [n]$. This isomorphism maps the labeling λ to one of the standard EL-labelings of the partition lattice [Bjö80,

Example 2.9]. In particular there is an unique increasing maximal chain from x to z .

Part 2: intervals of the form $[x, \hat{1}]$. Recursively define a maximal chain $c_{x\hat{1}}$ in $[x, \hat{1}]$ as follows. For $x \prec \hat{1}$, define $c_{x\hat{1}} = (x \prec \hat{1})$. Here notice that $\lambda(x, \hat{1}) = (1, 2)$. Suppose now that $x \prec \hat{1}$ is not covered by $\hat{1}$.

- Case 1: there exist unsigned normalized blocks $R \neq R'$ in x . Among all such pairs (R, R') , choose the (only) one for which $(r(R), r(R'))$ is lexicographically minimal. Let z be the Σ -labeled partition obtained from x by replacing $R \mid \bar{R} \mid R' \mid \bar{R}'$ with $R \cup R' \mid \bar{R} \cup \bar{R}'$. Then set $c_{x\hat{1}} = (x \prec c_{z\hat{1}})$. Notice that (x, z) is coherent, and $\lambda(x, z) = (0, r(R'))$.
- Case 2: no such pair (R, R') exists. Since x is not covered by $\hat{1}$, there exists a unique normalized unsigned block R in x . Let z be the Σ -labeled partition obtained from x by labeling $R \cup \bar{R}$ by σ , where

$$\sigma = \begin{cases} \min_{<} \Sigma(x) & \text{if } \Sigma(x) \neq \emptyset \\ \min_{<} \Sigma & \text{if } \Sigma(x) = \emptyset. \end{cases}$$

Then set $c_{x\hat{1}} = x \prec c_{z\hat{1}}$. Notice that (x, z) is of sign σ , and $\lambda(x, z) = (1, 1)$. In addition, notice that $z \prec \hat{1}$, so $c_{x\hat{1}} = (x \prec z \prec \hat{1})$.

We are going to prove that $c_{x\hat{1}}$ is the unique increasing maximal chain in $[x, \hat{1}]$, and that it is lexicographically minimal among all the maximal chains in $[x, \hat{1}]$.

- By construction, $c_{x\hat{1}}$ is increasing.
- Let c be any increasing maximal chain in $[x, \hat{1}]$. We want to show that $c = c_{x\hat{1}}$. Assume that x is not covered by $\hat{1}$, otherwise the chain is trivial. Since the last label of c is $(1, 2)$, c does not contain any non-coherent edges, and it contains exactly one signed edge. So c contains: first, a sequence of coherent edges, with labels of the form $(0, *)$; then, one signed edge labeled $(1, 1)$; finally, one edge ending in $\hat{1}$, labeled $(1, 2)$. Therefore there exists some $z \in c$ such that $c \cap [x, z]$ only consists of coherent edges, and $c \cap [z, \hat{1}]$ consists in one signed edge and one edge ending in $\hat{1}$. Here z is uniquely determined by x , and $c \cap [z, \hat{1}]$ is uniquely determined by the increasing property of c . The interval $[x, z]$ contains only coherent edges, thus by Part 1 it contains a unique increasing maximal chain which must coincide with $c \cap [x, z]$. Therefore there is an unique increasing maximal chain in $[x, \hat{1}]$ and $c = c_{x\hat{1}}$.
- In both Case 1 and Case 2 above, $\lambda(x, z) < \lambda(x, z')$ for every $z' \neq z$ that covers x in $[x, \hat{1}]$. Then $c_{x\hat{1}}$ is lexicographically minimal among the maximal chains in $[x, \hat{1}]$.

Part 3: intervals $[x, y]$ where y is of the form $y = R \mid \bar{R}$. Recursively define a maximal chain c_{xy} in $[x, y]$ as follows. For $x = y$, set $c_{yy} = (y)$. Assume now $x \prec y$.

- Case 1: there exist unsigned normalized blocks $T \neq T'$ in x such that $T \cup T'$ is contained in a block of y . Then choose the unique pair (T, T') of blocks of x such that $(r(T), r(T'))$ is lexicographically minimal. Let z be the partition obtained from x by replacing $T \mid \bar{T} \mid T' \mid \bar{T}'$ with $T \cup T' \mid \bar{T} \cup \bar{T}'$, and set $c_{xy} = (x \prec c_{zy})$. Notice that (x, z) is coherent, and $\lambda(x, z) = (0, r(T'))$.
- Case 2: no such pair (T, T') exists. Then $x \prec y$, and there exist unsigned normalized blocks $Q \neq Q'$ in x such that $Q \cup Q'$ coincides with R or \bar{R} . Set $c_{xy} = (x \prec y)$. Notice that (x, y) is non-coherent, and $\lambda(x, y) = (2, r)$, where $r = \min(r(Q), r(Q'))$.

We are going to prove that c_{xy} is the unique increasing maximal chain in $[x, y]$, and that it is lexicographically minimal among all the maximal chains in $[x, y]$.

- By construction, c_{xy} is increasing.
- Let c be any increasing maximal chain in $[x, y]$. We want to show that $c = c_{xy}$. The chains in $[x, y]$ have no signed edges. Since the labels of coherent edges are smaller than the labels of non-coherent edges, coherent edges precede non-coherent edges along c . So there exists an element $z \in c$ such that $c \cap [x, z]$ only consists of coherent edges, and $c \cap [z, y]$ only consists of non-coherent edges.

If the last edge of c is non-coherent, then its label must be $(2, r(R))$. This is also the minimum possible label for a non-coherent edge in the interval $[x, y]$. Therefore, since c is increasing, it contains at most one non-coherent edge.

Thus z is uniquely determined by x and y : if R is the union of the normalized blocks of x , then c cannot end with a non-coherent edge, so $z = y$; otherwise, $z = Q \mid \bar{Q} \mid Q' \mid \bar{Q}'$, where $Q \cup Q' = R$ and both Q and Q' are unions of normalized blocks of x . Also $c \cap [z, y]$ is uniquely determined.

Consider now the interval $[x, z]$. This interval contains only coherent edges, thus by Part 1 it contains a unique increasing maximal chain which must coincide with $c \cap [x, z]$. Therefore there is an unique increasing maximal chain in $[x, \hat{1}]$, and $c = c_{x\hat{1}}$.

- In both Case 1 and Case 2 above, $\lambda(x, z) < \lambda(x, z')$ for any $z' \neq z$ that covers x in the interval $[x, y]$. Then c_{xy} is lexicographically minimal among the maximal chains in $[x, y]$.

Part 4: intervals $[x, y]$ where y has only signed blocks or (unsigned) singleton blocks. Recursively define a maximal chain c_{xy} in $[x, y]$ as follows. For $x = y$, set $c_{yy} = (y)$. Assume now $x \prec y$.

- Case 1: there exist unsigned normalized blocks $R \neq R'$ of x such that $R \cup R'$ is contained in some block of y . Among all such pairs (R, R') , choose the (only) one for which $(r(R), r(R'))$ is lexicographically minimal. Let z be the Σ -labeled partition obtained from x by replacing $R \mid \bar{R} \mid R' \mid \bar{R}'$ with $R \cup R' \mid \bar{R} \cup \bar{R}'$. Set $c_{xy} = (x \prec c_{zy})$. Notice that (x, z) is coherent, and $\lambda(x, z) = (0, r(R'))$.
- Case 2: no such pair (R, R') exists. Then there are at most $|\Sigma(y)|$ elements $z \in [x, y]$ that cover x , since for every sign $\sigma \in \Sigma(y)$ there is at most one element z such that (x, z) is of sign σ . If possible, choose z such that the sign of (x, z) equals $\min \{\sigma \in \Sigma(x) \mid x_\sigma \subsetneq y_\sigma\}$. Otherwise, choose z such that the sign of (x, z) equals $\min \{\sigma \in \Sigma(y) \mid x_\sigma \subsetneq y_\sigma\}$. Set $c_{xy} = (x \prec c_{zy})$. Notice that (x, z) is a signed edge, and $\lambda(x, z)$ is of the form $(1, *)$.

We are going to prove that c_{xy} is the unique increasing maximal chain in $[x, y]$, and that it is lexicographically minimal among all the maximal chains in $[x, y]$.

- By construction, c_{xy} is increasing.
- Let c be any increasing maximal chain in $[x, y]$. We want to show that $c = c_{xy}$. Since y contains no unsigned block, the last edge of c must be signed. In particular, it must have a label of the form $(1, *)$. Then an increasing chain in $[x, y]$ can only contain coherent edges and signed edges. In addition, coherent edges precede signed edges along c . Therefore there exists an element $z \in c$ such that $c \cap [x, z]$ only consists of coherent edges, and $c \cap [z, y]$ only consists of signed edges.

It is clear that every maximal chain in $[x, y]$ contains at least

$$t = |\{\sigma \in \Sigma(y) \mid x_\sigma \subsetneq y_\sigma\}|$$

signed edges. Suppose now that a maximal chain d of $[x, y]$ contains more than t signed edges. So d contains at least two edges of the same sign τ , and one of the following cases occurs — if d has more than two edges of sign τ , consider the first two such edges.

- $\tau \in \Sigma(x)$ and the two edges of sign τ are separated only by edges of sign in $\Sigma(x)$. Then d has two edges with the same label, and therefore it is not increasing.
- $\tau \in \Sigma(x)$ and the two edges of sign τ are separated only by edges of sign $\sigma \in \Sigma(y) \setminus \Sigma(x)$ such that $\tau < \sigma$. Then the two edges of sign τ have the same label, and d is not increasing.

- $\tau \in \Sigma(x)$ and the two edges of sign τ are separated by some edge of sign $\sigma \in \Sigma(y) \setminus \Sigma(x)$ such that $\sigma < \tau$. Then the last such edge has a label greater than or equal to the label of the second edge of sign τ , and d is not increasing.
- $\tau \in \Sigma(y) \setminus \Sigma(x)$. Then the second edge of sign τ has a label which is smaller than or equal to the label of the first edge of sign τ , and d is not increasing.

In any case, d is not increasing. So every increasing maximal chain in $[x, y]$ contains exactly t signed edges, and all these edges have a different sign.

Therefore z is uniquely determined by x and y . The only way for $c \cap [z, y]$ to be increasing is to contain first all the edges of sign $\sigma \in \Sigma(x)$ in increasing order, and then all the edges of sign $\sigma \in \Sigma(y) \setminus \Sigma(x)$ in increasing order. Then $c \cap [z, y]$ is also uniquely determined.

Consider now the interval $[x, z]$, which contains only coherent edges. By Part 1, this interval has a unique increasing maximal chain which must coincide with $c \cap [x, z]$. In particular, there is an unique increasing maximal chain in $[x, y]$, and $c = c_{xy}$.

- In both Case 1 and Case 2 above, $\lambda(x, z) < \lambda(x, z')$ for any $z' \neq z$ that covers x in the interval $[x, y]$. Then c_{xy} is lexicographically minimal among the maximal chains in $[x, y]$.

Part 5: general intervals $[x, z]$ with $z \neq \hat{1}$. Given any set of disjoint (signed or unsigned) blocks X_1, \dots, X_t , denote by $\{\{X_1, \dots, X_t\}\}$ the Σ -labeled partition whose blocks are X_1, \dots, X_t and a singleton block for every element of $[[n]] \setminus (X_1 \cup \dots \cup X_t)$. Let

$$z = z_{\sigma_1} \mid \dots \mid z_{\sigma_t} \mid R_1 \mid \bar{R}_1 \mid \dots \mid R_k \mid \bar{R}_k.$$

Consider the Σ -labeled partition $x_0 = \{\{B \in x \mid B \subseteq z_\sigma \text{ for some } \sigma \in \Sigma(z)\}\}$, and define the following intervals:

$$J(0) = [x_0, \{\{z_{\sigma_1}, \dots, z_{\sigma_t}\}\}]$$

$$J(i) = [\{\{B \in x : B \subseteq R_i \cup \bar{R}_i\}\}, \{\{R_i, \bar{R}_i\}\}] \text{ for } i = 1, \dots, k.$$

There is a poset isomorphism

$$J(0) \times \dots \times J(k) \rightarrow [x, z]$$

mapping every $(k+1)$ -tuple of partitions on the left-hand side to their common refinement.

In Parts 3 and 4 we proved that the labeling λ on each $J(i)$ is an EL-labeling. If we denote by Λ_i the set of labels used in $J(i)$, we see that $\Lambda_i \cap \Lambda_j = \emptyset$ whenever $i \neq j$. Thus, the lexicographic order on $\Lambda_0 \cup \dots \cup \Lambda_k$ is a shuffle of the lexicographic orders of the Λ_i 's. The edge labeling induced on $[x, z]$ is equal to λ . By Theorem 1.5.3, this is then an EL-labeling of $[x, z]$. \square

7.5 EL-SHELLABILITY OF POSETS OF LAYERS

Using Theorem 7.4.2, we are now able to prove that the posets of layers of arrangements defined by root systems are EL-shellable.

Theorem 7.5.1 (EL-shellability of subposets). For any set of signs Σ with a distinguished element $\bar{\sigma} \in \Sigma$, the posets of Σ -labeled partitions of type A_{n-1} , B_n , C_n , and D_n are EL-shellable.

Proof. For type C_n , it is a direct consequence of Theorem 7.4.2. For type A_{n-1} , the poset of layers is isomorphic to a classical partition lattice, which is EL-shellable because it is a geometric lattice.

Denote by \mathcal{P} a poset of Σ -labeled partitions of type B_n or D_n , with a top element $\hat{1}$ added. Then, \mathcal{P} is a ranked subposet of $\Pi_{n,\Sigma}$ with rank function induced by the rank function of the ambient poset. With the idea of applying Lemma 1.5.4, consider two elements $x \preceq z$ in \mathcal{P} , and denote by

$$x \prec x_1 \prec \cdots \prec \underbrace{x_{i-1} \prec x_i}_{e_i} \prec \cdots \prec z$$

the unique increasing chain in $[x, z] \subseteq \Pi_{n,\Sigma}$. We have to show that this chain is contained in \mathcal{P} . Suppose by contradiction that this is not the case, and let i be the minimal index such that $x_i \notin \mathcal{P}$. Then there is a sign $\sigma \in \Sigma(z) \setminus \Sigma(x_{i-1})$ such that $|(x_i)_\sigma| = 2$. In particular, e_i is a signed edge. This implies that $z_\sigma \neq \emptyset$, but since $z \in \mathcal{P}$ we have $z_\sigma \supseteq (x_i)_\sigma$. Then there must be another edge e_j of sign σ with $j > i$. Therefore $\lambda(e_j) \leq \lambda(e_i)$, which is a contradiction. \square

Corollary 7.5.2 (EL-shellability of posets of layers). Posets of layers of linear, toric, and elliptic arrangements defined by root systems are EL-shellable. In particular, $\Delta(\tilde{\mathcal{L}}(\mathcal{A}))$ is homotopy equivalent to a wedge of $|T_{\mathcal{A}}(1, 0)|$ spheres if $\mathcal{L}(\mathcal{A})$ is bounded (this happens in the case A and for linear arrangements), and to a wedge of $|T_{\mathcal{A}}(0, 0)|$ spheres otherwise.

Proof. The first part follows immediately from Theorems 7.3.1 and 7.5.1. The second part follows from Theorem 1.4.5 and from the final remark on Tutte polynomials in Section 2.7. The implication is direct if $\mathcal{L}(\mathcal{A})$ is bounded. Otherwise, we complete $\mathcal{L}(\mathcal{A})$ by adding a maximal element and we use the identity $\chi_P(0) = -\chi_{P \setminus \{\hat{1}\}}(1)$, valid for every bounded poset P . \square

Remark 7.5.3. Corollary 7.5.2 describes the homotopy type of posets of layers in terms of the Tutte polynomial of the arrangement. This already allows to fill in the leftmost column of Table 7.1 and, in principle, also the middle column (see Remark 7.3.4). In the elliptic case, no explicit form for the Tutte polynomials is known to us. In the following section we address the problem of determining the homotopy type of these posets from another (unified) perspective.

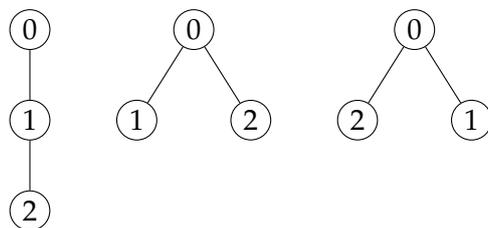


Figure 7.1: All different increasing ordered trees on 3 nodes. The second and the third tree differ in the total order of the children of the root.

7.6 HOMOTOPY TYPE

In order to determine the homotopy type of posets of labeled partitions, we need to count maximal decreasing chains in our EL-labeled posets (see Section 1.5). To this end, we first introduce *increasing ordered trees*. Such trees appeared in the literature under various names (e.g. heap-ordered trees [CN94], ordered trees with no inversions [GSY95], simple drawings of rooted plane trees [Kla97]).

Definition 7.6.1. An *increasing ordered tree* is a rooted tree, with nodes in bijection with a finite subset $L \subset \mathbb{N}$ of labels, such that:

- each path from the root to any leaf has increasing labels (in particular, the root is labeled by the minimal element of L);
- for each node, a total order of its children is specified.

If L is not given, it is assumed to be $\{0, 1, \dots, n\}$ for some integer $n \geq 0$.

See Figure 7.1 for a few examples. The number of increasing ordered trees on $n + 1$ nodes is $(2n - 1)!!$. This is a classical result which appeared several times in the literature [CN94, GSY95, Kla97]. A simple proof by induction is as follows: given an increasing ordered tree on n nodes (labeled $0, \dots, n - 1$), there are $2n - 1$ ways to append an additional node labeled n , to form an increasing ordered tree on $n + 1$ nodes.

We are now ready to state and prove the general result about the number of decreasing maximal chains.

Theorem 7.6.2 (Decreasing chains). The number of decreasing maximal chains from $\hat{0}$ to $\hat{1}$ in the subposet of $\Pi_{n,\Sigma}$ of type B_n , C_n , or D_n , is

$$\sum_{\pi = \pi_1 | \dots | \pi_k \in \Pi_n} f(\pi) \cdot (2|\pi_1| - 3)!! \cdots (2|\pi_k| - 3)!! \quad (7.2)$$

Here Π_n is the poset of standard partitions of $[n]$, and $f: \Pi_n \rightarrow \mathbb{N}$ depends on the type (B_n , C_n , or D_n) and on $m = |\Sigma|$. In order to define $f(\pi)$, denote by k the number of blocks of π , and by s the number of nonsingleton blocks of π .

Type C_n : $f(\pi) = \frac{(k+m-2)!}{(m-2)!}$. For $m = 1$, set $f(\pi) = 0$.

Type D_n : $f(\pi) = \begin{cases} \sum_{r=2}^{\min(m,s)} \binom{m}{r} \frac{(k-2)!}{(r-2)!} \frac{s!}{(s-r)!} & \text{if } \pi \neq [n] \\ m-1 & \text{if } \pi = [n]. \end{cases}$

Type B_n : $f(\pi) = \begin{cases} \sum_{r=2}^{\min(m,s+1)} \binom{m-1}{r-1} \frac{(k-2)!}{(r-2)!} (k-r+1) \frac{s!}{(s-r+1)!} & \text{if } \pi \neq [n] \\ + \sum_{r=2}^{\min(m-1,s)} \binom{m-1}{r} \frac{(k-2)!}{(r-2)!} \frac{s!}{(s-r)!} & \\ m-1 & \text{if } \pi = [n]. \end{cases}$

Proof. Let \mathcal{D} be the set of decreasing maximal chains from $\hat{0}$ to $\hat{1}$. When read bottom-to-top, chains $c \in \mathcal{D}$ consist of:

- a sequence of non-coherent edges, labeled $(2, *)$;
- then, a sequence of signed edges, labeled $(1, *)$;
- finally, one last edge labeled $(1, 2)$.

Define a projection $\eta: \mathcal{D} \rightarrow \Pi_{n,\emptyset} \subseteq \Pi_{n,\Sigma}$, mapping a chain $c \in \mathcal{D}$ to the (unique) element $z \in c$ which comes after all the non-coherent edges and before all the signed edges. In addition, let $\rho: \Pi_{n,\emptyset} \rightarrow \Pi_n$ be the natural order-preserving projection that maps a partition $z = R_1 | \bar{R}_1 | \cdots | R_k | \bar{R}_k \in \Pi_{n,\emptyset}$ to the partition $\pi = \pi_1 | \cdots | \pi_k \in \Pi_n$ with blocks $\pi_i = (R_i \cup \bar{R}_i) \cap [n]$.

Label the edges of Π_n following the “non-coherent” rule: if $\alpha \prec \beta$ in Π_n , and β is obtained from α by merging the two blocks α_i and α_j , then label (α, β) by $\min(\alpha_i \cup \alpha_j)$. This is not an EL-labeling of Π_n , but it is the (second component of the) image of our EL-labeling of $\Pi_{n,\Sigma}$ through the map ρ , restricted to non-coherent edges of $\Pi_{n,\emptyset} \subseteq \Pi_{n,\Sigma}$.

Both maps η and ρ are clearly surjective. We claim that, for each partition $\pi = \pi_1 | \cdots | \pi_k \in \Pi_n$, we have

$$|(\rho \circ \eta)^{-1}(\pi)| = f(\pi) \cdot (2|\pi_1| - 3)!! \cdots (2|\pi_k| - 3)!!$$

We prove this through the following steps.

1. The number of decreasing maximal chains of $[\hat{0}, \pi] \subseteq \Pi_n$ is

$$(2|\pi_1| - 3)!! \cdots (2|\pi_k| - 3)!!$$

2. In the preimage under ρ of every decreasing maximal chain of $[\hat{0}, \pi] \subseteq \Pi_n$, there is exactly one decreasing chain of $\Pi_{n,\emptyset}$ with all non-coherent edges.

3. For every $z \in \rho^{-1}(\pi)$, the number of decreasing maximal chains of $[z, \hat{1}]$ without unsigned edges is $f(\pi)$. Notice that the interval $[z, \hat{1}]$ is different in each subsubset of $\Pi_{n,\Sigma}$, depending on the type (B_n , C_n , or D_n); the definition of f changes accordingly.

These three steps prove the above claim, which completes the proof of this theorem. We now prove each of the three steps.

1. The interval $[\hat{0}, \pi]$ is a product of intervals $[\hat{0}, \pi_1], \dots, [\hat{0}, \pi_k]$, and the labeling splits accordingly. Without loss of generality, we can then assume that π consists of a single block $\pi_1 \cong \{1, \dots, \ell\}$. Decreasing maximal chains c of $[\hat{0}, \pi]$ are in one-to-one correspondence with increasing ordered trees T on nodes $\{1, \dots, \ell\}$: a node j is a direct child of a node $i < j$ in T if and only if the chain c features a merge of blocks R and R' with $\min(R) = i$ and $\min(R') = j$; the set of direct children of a node is ordered according to the sequence of the corresponding merges in c . The number of such trees is $(2\ell - 3)!!$, as discussed at the beginning of this section.
2. Given a decreasing maximal chain c of $[\hat{0}, \pi] \subseteq \Pi_n$, a lift to $\Pi_{n,\emptyset}$ can be constructed explicitly by induction, starting from $\hat{0}$ and parsing the chain in increasing order. In fact, for every edge $\alpha \prec \beta$ in Π_n and for every choice of $x \in \rho^{-1}(\alpha)$, there are two ways to lift β to some $y \succ x$, and in exactly one of these two cases the resulting edge (x, y) is non-coherent.
3. Such a chain is uniquely determined by an ordering of the blocks π_1, \dots, π_k of π (which determines the order of the merges), and by a decreasing sequence of k labels

$$(1, m) \geq (1, a_1) \geq \dots \geq (1, a_k) \geq (1, 2) \quad (7.3)$$

to be assigned to the edges (the sequence of labels uniquely determines the sequence of signs). Any ordering of the blocks of π and any decreasing sequence of labels gives rise to a valid chain, provided that (in types B_n and D_n) no unadmissible block $i\bar{i}_\sigma$ is created. Given a decreasing sequence of labels, we call such an ordering of the blocks of π a *valid ordering*. In type C_n , any ordering of the blocks of π is always valid, regardless of the sequence of labels. Notice that for $m = 1$ there are no decreasing sequences of labels: in this case $f(\pi) = 0$.

- Consider type C_n , so that all signed blocks are admissible. Then there are $k!$ possible orderings of the blocks, and $\binom{k+m-2}{k}$ decreasing sequences of k labels. So we get

$$f(\pi) = k! \cdot \binom{k+m-2}{k} = \frac{(k+m-2)!}{(m-2)!}.$$

- Consider now type D_n . The sequence of signs associated to a decreasing sequence of labels as in (7.3) contains exactly r different signs if and only if $a_r \geq r$ and $a_{r+1} \leq r$. Therefore, for a fixed $r \in \{1, \dots, m\}$, the sequences of labels which give rise to exactly r different signs are those of the form

$$\begin{aligned} (1, m) &\geq (1, a_1) \geq \dots \geq (1, a_r) \geq (1, r) \\ &\geq (1, a_{r+1}) \geq \dots \geq (1, a_k) \geq (1, 2). \end{aligned}$$

The number of such sequences is $\binom{m}{r} \binom{k-2}{r-2}$ for $r \geq 2$. For $r = 1$, there are $m - 1$ sequences if $k = 1$, and 0 otherwise.

If $k = 1$ (i.e. $\pi = [n]$), exactly one sign is used and we get $f([n]) = m - 1$. Suppose from now on $k \geq 2$, i.e. $\pi \neq [n]$. We have to compute the number of valid orderings of the k blocks of π . Recall that s is the number of nonsingleton blocks of π . Each time a sign appears for the first time in the chain, a nonsingleton block must be signed. Then, for a fixed number $r \in \{2, \dots, m\}$ of signs appearing in the chain, the number of valid orderings of the k blocks is $s(s-1) \dots (s-r+1) \cdot (k-r)!$. Therefore

$$\begin{aligned} f(\pi) &= \sum_{r=2}^{\min(m,s)} \binom{m}{r} \binom{k-2}{r-2} \frac{s!}{(s-r)!} (k-r)! \\ &= \sum_{r=2}^{\min(m,s)} \binom{m}{r} \frac{(k-2)!}{(r-2)!} \frac{s!}{(s-r)!}. \end{aligned}$$

- Finally consider type B_n , where we have a distinguished sign $\bar{\sigma}$ that can admit doubleton blocks. It is convenient to assume that $\bar{\sigma}$ is the maximal element of Σ with respect to the total order of Definition 7.4.1. Then a sequence of labels $(1, m) \geq (1, a_1) \geq \dots \geq (1, a_k) \geq (1, 2)$ gives rise to a sequence of signs containing $\bar{\sigma}$ if and only if $a_1 = m$. Both cases $a_1 = m$ and $a_1 \leq m - 1$ can be treated as in type D_n . \square

Linear arrangements

In the case of linear arrangements ($m = 1$), the number of maximal decreasing chains is 0. This happens because $\Pi_{n, \{\sigma\}} \setminus \{\hat{1}\}$ still has a unique maximal element $[n]_\sigma$. Therefore one is interested in counting the decreasing chains from $\hat{0}$ to $[n]_\sigma$.

Theorem 7.6.3. The number of maximal decreasing chains from $\hat{0}$ to $[n]_\sigma$, in the subposet of $\Pi_{n, \{\sigma\}}$ of type B_n , C_n , or D_n , is given by (7.2), with the following definition of $f: \Pi_n \rightarrow \mathbb{N}$.

Type $B_n = C_n$: $f(\pi) = k!$

Type D_n : $f(\pi) = s \cdot (k - 1)!$

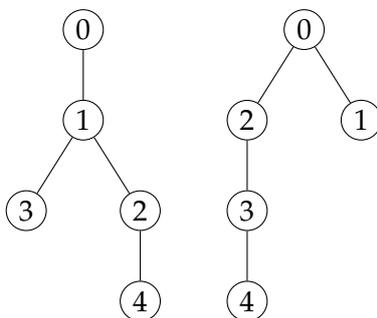


Figure 7.2: These trees are 1-flourishing but not 2-flourishing.

Proof. It is completely analogous to the proof of Theorem 7.6.2. \square

We want to give a graph-theoretic interpretation of Formula (7.2) for these two definitions of $f: \Pi_n \rightarrow \mathbb{N}$, in order to derive the well-known closed formulas of Table 7.1 for the linear types $B_n = C_n$ and D_n .

Proposition 7.6.4. Formula (7.2) with $f(\pi) = k!$ counts the increasing ordered trees on $n + 1$ nodes. Therefore the number of decreasing chains from $\hat{0}$ to $[n]_\sigma$ in the linear type $B_n = C_n$ is $(2n - 1)!!$

Proof. Consider an increasing ordered tree on $n + 1$ nodes, labeled $0, 1, \dots, n$. The set of subtrees corresponding to the children of the root is a partition of $\{1, \dots, n\}$. For a fixed partition $\pi \in \Pi_n$, the possible trees that induce this partition can be recovered as follows: order the blocks of π in $k!$ ways; for each block π_i , construct an increasing ordered tree with nodes labeled by elements of π_i , in $(2|\pi_i| - 3)!!$ ways. Summing over the partitions $\pi \in \Pi_n$, we obtain Formula (7.2). \square

An analogous interpretation can be given for the linear type D_n , provided that we introduce the following class of increasing ordered trees.

Definition 7.6.5. Let $r \geq 0$ be an integer. An increasing ordered tree is r -flourishing if the root has at least r children, and none of the first r children of the root is a leaf. See Figure 7.2 for some examples.

Lemma 7.6.6. The number of r -flourishing trees on $n + 1$ nodes is given by

$$(2n - 2r - 1)!! \cdot (n - r)(n - r - 1) \cdots (n - 2r + 1)$$

for $n \geq 2r$, and 0 otherwise.

Proof. Let $v(n, r)$ be the number of r -flourishing trees on $n + 1$ nodes. We are going to prove the following recursive relation:

$$v(n, r) = v(n - 1, r) \cdot (2n - r - 1) + v(n - 2, r - 1) \cdot (n - 1)r.$$

- The first summand counts the r -flourishing trees obtained by appending an additional node n to some r -flourishing tree on n nodes. The node n cannot be appended as one of the first r children of the root, so $2n - r - 1$ possible positions remain.
- The second summand counts the r -flourishing trees that would not be r -flourishing after removing the node n . Node n must be the only child of some other node i , which must be one of the first r children of the root. There are $n - 1$ choices for i , and r choices for its position. Once the nodes n and i are removed, a $(r - 1)$ -flourishing tree on n nodes remains.

The formula given in the statement holds for $r = 0$ or $n = 0$, and a straightforward computation shows that it satisfies the recursive relation. \square

Similarly to Proposition 7.6.4, the number of decreasing chains for the linear type D_n has the following graph-theoretic interpretation.

Proposition 7.6.7. Formula (7.2) with $f(\pi) = s \cdot (k - 1)!$ counts the increasing ordered 1-flourishing trees on $n + 1$ nodes. Therefore the number of maximal decreasing chains from $\hat{0}$ to $[n]_\sigma$ in the linear type D_n is $(2n - 3)!! \cdot (n - 1)$.

Proof. The first part is analogous to the proof of Proposition 7.6.4. The second part then follows from Lemma 7.6.6 for $r = 1$. \square

Toric arrangements

In the toric case ($m = 2$), the formulas of Theorem 7.6.2 are as follows.

Type C_n : $f(\pi) = k!$

Type B_n : $f(\pi) = s \cdot (k - 1)!$

Type D_n : $f(\pi) = \begin{cases} s(s - 1) \cdot (k - 2)! & \text{if } \pi \neq [n] \\ 1 & \text{if } \pi = [n]. \end{cases}$

In particular, the number of decreasing chains in the toric type C_n is the same as in the linear type C_n (or B_n), and the number of decreasing chains in the toric type B_n is the same as in the linear type D_n .

A closed formula for the toric type D_n can also be found, similarly to the linear type D_n . One only has to separately take care of the partition $\pi = [n]$, which contributes by $(2n - 3)!!$ to the total sum.

Proposition 7.6.8. Formula (7.2) with $f(\pi) = s(s - 1) \cdot (k - 2)!$ counts the increasing ordered 2-flourishing trees on $n + 1$ nodes. Therefore the number of maximal decreasing chains from $\hat{0}$ to $\hat{1}$ in the toric type D_n is

$$(2n - 5)!! \cdot (n - 2)(n - 3) + (2n - 3)!! = (2n - 5)!! \cdot (n^2 - 3n + 3).$$

Proof. The first part is analogous to the proof of Proposition 7.6.4. Then the second part follows from Lemma 7.6.6 for $r = 2$. \square

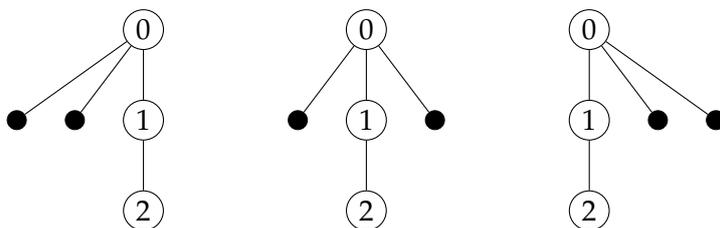


Figure 7.3: All different 2-blooming trees constructed from the leftmost increasing ordered tree of Figure 7.1. These are 3 of the 15 different 2-blooming trees on 3 (labeled) nodes. Blooms are shown as smaller black (unlabeled) nodes.

Elliptic arrangements

In the elliptic case ($m = 4$), we are going to derive a closed formula only for type C_n . In order to do so, we introduce a variant of increasing ordered trees.

Definition 7.6.9. Let $q \geq 0$ be an integer. A q -blooming tree on $n + 1$ nodes is an increasing ordered tree on $n + 1$ nodes, with q extra indistinguishable unlabeled nodes (called *blooms*) appended to the root. The only thing that matters about blooms is their position in the total order of the children of the root. See Figure 7.3 for some examples.

Lemma 7.6.10. The number of q -blooming trees on $n + 1$ nodes is

$$\frac{(2n + q - 1)!!}{(q - 1)!!}.$$

Proof. The proof is by induction on n . For $n = 0$ there is only one q -blooming tree, consisting of the root with q blooms attached to it. Given a q -blooming tree on $n \geq 1$ nodes (labeled $0, \dots, n - 1$), there are exactly $2n + q - 1$ positions where an additional node n can be attached in order to obtain a q -blooming tree on $n + 1$ nodes. Every q -blooming tree on $n + 1$ nodes is obtained exactly once from some q -blooming tree on n nodes. \square

Proposition 7.6.11. Formula (7.2) with $f(\pi) = \frac{(k+m-2)!}{(m-2)!}$ counts the increasing ordered $(m - 2)$ -blooming trees on $n + 1$ nodes. Therefore the number of decreasing chains from $\hat{0}$ to $\hat{1}$ in the elliptic type C_n is $(2n + 1)!!$

Proof. The first part is analogous to the proof of Proposition 7.6.4. Then the second part follows from Lemma 7.6.10 for $q = 2$. \square

Remark 7.6.12. A closed formula for r -flourishing q -blooming trees on $n + 1$ nodes (if such a formula exists at all) would yield a closed formula for the number of decreasing chains also for the elliptic types B_n and D_n . Here by

7. SHELLABILITY OF ARRANGEMENTS DEFINED BY ROOT SYSTEMS

an r -flourishing q -blooming tree we mean a q -blooming tree whose r leftmost children of the root are neither leaves nor blooms. For example, among the three 2-blooming trees in Figure 7.3 only the rightmost one is 1-flourishing.

Shellability of generalized Dowling posets

8.1 INTRODUCTION

In this chapter we see how the construction of Chapter 7 fits into a more general picture. The material is based on [Pao18b].

In a recent contribution to the study of orbit configuration spaces [BG18], Bibby and Gadish introduced a class of posets $\mathcal{D}_n(G, S)$ which they called *S-Dowling posets*. Here n is a positive integer, S is a finite set, and G is a finite group acting on S . These posets arise as posets of layers of arrangements $\mathcal{A}_n(G, X)$ of “singular subspaces” in X^n , where X is a space with a G -action. They generalize both Dowling lattices [Dow73] (which are obtained for $|S| = 1$) and posets of layers of linear, toric, and elliptic arrangements defined by root systems of type C_n [Bib18] (obtained if $G = \mathbb{Z}_2$ acts trivially on S , and $|S| \in \{1, 2, 4\}$).

Dowling lattices have long been known to be shellable [Got98], and in Chapter 7 we established shellability of posets of layers of arrangements defined by root systems (this was a joint work with Emanuele Delucchi and Noriane Girard [DGP17]). A natural question posed in [BG18] is to prove shellability for the *S-Dowling posets* $\mathcal{D}_n(G, S)$. We solve this conjecture in a positive way.

Theorem 8.3.5. The poset $\mathcal{D}_n(G, S) \cup \{\hat{1}\}$ is EL-shellable.

The order complex of a shellable poset is homotopy equivalent to a wedge of spheres. Once shellability is proved, the number of spheres in the order complex of $\mathcal{D}_n(G, S) \setminus \{\hat{0}\}$ can be retrieved from the characteristic polynomial, which is computed in [BG18, Theorem 2.5.2]. In Section 8.4 we give an

independent combinatorial computation of this number, based on counting rooted trees.

Theorem 8.4.3. The order complex of the poset $\mathcal{D}_n(G, S) \setminus \{\hat{0}\}$ is homotopy equivalent to a wedge of

$$(-1)^\epsilon \prod_{i=0}^{n-1} (|S| - 1 + |G|i)$$

$(n - 1 - \epsilon)$ -dimensional spheres, except for the empty poset $\bar{\mathcal{D}}_1(\{e\}, \emptyset)$. Here $\epsilon = 0$ for $S \neq \emptyset$, and $\epsilon = 1$ for $S = \emptyset$.

In the study of the posets of layers of invariant arrangements, Bibby and Gadish also introduced subposets $\mathcal{P}_n(G, S, T) \subseteq \mathcal{D}_n(G, S)$ corresponding to any G -invariant subset $T \subseteq S$. When $G = \mathbb{Z}_2$ acts trivially on S , suitable choices of T yield posets of layers of arrangements defined by root systems of type B_n and D_n [Bib18], which were proved to be shellable in Chapter 7. Therefore it is natural to ask if the subposets $\mathcal{P}_n(G, S, T)$ are shellable in general. In Section 8.5 we exhibit a family of counterexamples, obtained when all the elements of S have a trivial G -stabilizer. However, we prove that $\mathcal{P}_n(G, S, T)$ is shellable if the G -action on $S \setminus T$ is trivial.

Theorem 8.5.8. If the G -action on $S \setminus T$ is trivial, the poset $\mathcal{P}_n(G, S, T) \cup \{\hat{1}\}$ is EL-shellable.

Finally, we determine the homotopy type of a larger class of subposets.

Theorem 8.5.11. Let $n \geq 2$ and $|S| \geq 1$. Suppose that all the G -orbits in $S \setminus T$ either have cardinality 1, or have a trivial stabilizer. Then the poset $\mathcal{P}_n(G, S, T) \setminus \{\hat{0}\}$ is homotopy equivalent to a wedge of d -dimensional spheres with

$$d = \begin{cases} n - 1 & \text{if } T \neq \emptyset \text{ or at least one } G\text{-orbit is trivial} \\ n - 2 & \text{otherwise.} \end{cases}$$

8.2 GENERALIZED DOWLING POSETS

The definition of the poset $(\mathcal{D}_n(G, S), \preceq)$ is as follows [BG18, Section 2]. Let $[n] = \{1, 2, \dots, n\}$. A *partial G -partition* is a partition $\beta = \{B_1, \dots, B_\ell\}$ of the subset $\cup B_i \subseteq [n]$ together with functions $b_i: B_i \rightarrow G$ defined up to the following equivalence relation: $b_i \sim b'_i$ if $b_i = b'_i g$ for some $g \in G$. The functions b_i can be regarded as *projectivized G -colorings*. Define the *zero block* of β as $Z = [n] \setminus \cup B_i$. Then $\mathcal{D}_n(G, S)$ is the set of partial G -partitions β of $[n]$ together with an S -coloring of its zero block, i.e. a function $z: Z \rightarrow S$.

Following the conventions of [BG18], we use an uppercase letter B for a subset of $[n]$, the corresponding lowercase letter for the function $b: B \rightarrow G$,

and \tilde{B} for the data (B, \bar{b}) where \bar{b} is the equivalence class of $b: B \rightarrow G$. Then elements of $\mathcal{D}_n(G, S)$ take the form $(\tilde{\beta}, z)$, where $\tilde{\beta} = \{\tilde{B}_1, \dots, \tilde{B}_\ell\}$ and $z: Z \rightarrow S$ is the S -coloring of the zero block.

The set $\mathcal{D}_n(G, S)$ is partially ordered by covering relations (for which we use the symbol \prec), given by either merging two blocks or coloring one by S :

(merge) $(\tilde{\beta} \cup \{\tilde{A}, \tilde{B}\}, z) \prec (\tilde{\beta} \cup \{\tilde{C}\}, z)$ where $C = A \cup B$ and $c = a \cup bg$ for some $g \in G$;

(color) $(\tilde{\beta} \cup \{\tilde{B}\}, z) \prec (\tilde{\beta}, z')$ where z' is an extension of z to $Z' = B \cup Z$ such that $z'|_B$ is a composition

$$B \xrightarrow{b} G \xrightarrow{f} S$$

for some G -equivariant function f . Since f is uniquely determined by $s = f(e) \in S$ (where e is the identity element of G), we can equivalently say that $z'(i) = b(i) \cdot s$ for all $i \in B$.

The poset $\mathcal{D}_n(G, S)$ is ranked by the rank function $\text{rk}((\tilde{\beta}, z)) = n - |\tilde{\beta}|$. For $S = \emptyset$, the zero blocks are always empty. Therefore the rank of $\mathcal{D}_n(G, S)$ is $n - \epsilon$, where

$$\epsilon = \begin{cases} 0 & \text{if } S \neq \emptyset \\ 1 & \text{if } S = \emptyset. \end{cases}$$

An element $(\tilde{\beta}, z) \in \mathcal{D}_n(G, S)$ will be written also as in the following example:

$$[1_{g_1} 3_{g_3} \mid 2_{g_2} 4_{g_4} 6_{g_6} \parallel 5_{s_5} 7_{s_7}]$$

denotes the partial set partition $[13 \mid 246]$ with projectivized G -colorings $[g_1 : g_3]$ and $[g_2 : g_4 : g_6]$, and zero block $\{5, 7\}$ colored by $5 \mapsto s_5$ and $7 \mapsto s_7$.

Following [BG18, Section 3.4], we also introduce a subposet $\mathcal{P}_n(G, S, T) \subseteq \mathcal{D}_n(G, S)$ for any G -invariant subset $T \subseteq S$:

$$\mathcal{P}_n(G, S, T) = \{(\tilde{\beta}, z) \in \mathcal{D}_n(G, S) : |z^{-1}(O)| \neq 1 \text{ for all } G\text{-orbits } O \subseteq S \setminus T\}.$$

This subposet arises as the poset of layers of a certain invariant subarrangement $\mathcal{A}_n(G, X; T) \subseteq \mathcal{A}_n(G, X)$. It is ranked, with rank function induced by $\mathcal{D}_n(G, S)$, and $\text{rk } \mathcal{P}_n(G, S, T) = \text{rk } \mathcal{D}_n(G, S) = n - \epsilon$.

8.3 EL-SHELLABILITY OF GENERALIZED DOWLING POSETS

The poset $\mathcal{D}_n(G, S)$ contains a bottom element $\hat{0}$ but it is usually not bounded from above. We therefore introduce the bounded poset $\hat{\mathcal{D}}_n(G, S) = \mathcal{D}_n(G, S) \sqcup \{\hat{1}\}$ with $x \prec \hat{1}$ for all $x \in \mathcal{D}_n(G, S)$. In this section we are going to prove

that $\hat{\mathcal{D}}_n(G, S)$ is EL-shellable for every positive integer n , finite set S and finite group G acting on S . This solves [BG18, Conjecture 2.7.1]. The construction of the EL-labeling takes ideas from (and generalizes) the EL-labeling of Section 7.4.

Definition 8.3.1. Consider an edge $(x, y) \in \mathcal{E} = \mathcal{E}(\hat{\mathcal{D}}_n(G, S))$, with $y \neq \hat{1}$.

- Suppose that (x, y) is of type “merge”, i.e. $x = (\tilde{\beta} \cup \{\tilde{A}, \tilde{B}\}, z)$ and $y = (\tilde{\beta} \cup \{\tilde{C}\}, z)$, where $C = A \cup B$ and $c = a \cup bg$. We say that (x, y) is *coherent* if $c(\min A) = c(\min B)$, and *non-coherent* otherwise. If (x, y) is non-coherent, define $\alpha(x, y) \in G \setminus \{e\}$ as

$$\alpha(x, y) = \begin{cases} c(\min B) \cdot c(\min A)^{-1} & \text{if } \min A < \min B \\ c(\min A) \cdot c(\min B)^{-1} & \text{otherwise.} \end{cases}$$

Notice that this definition only depends on the \sim -equivalence class of c . The definition of α is motivated as follows: if c is normalized by setting $c(\min C) = e$, then $\{c(\min A), c(\min B)\} = \{e, \alpha(x, y)\}$.

- Suppose that (x, y) is of type “color”, i.e. $x = (\tilde{\beta} \cup \{\tilde{B}\}, z)$ and $y = (\tilde{\beta}, z')$. Then we say that (x, y) is *colored*, and its color is $z'(\min B)$.

Remark 8.3.2. The previous definition is a generalization of the one given in Section 7.4, with “signed” replaced by “colored”.

Definition 8.3.3 (Edge labeling of $\hat{\mathcal{D}}_n(G, S)$). Fix any total order $s_1 < s_2 < \dots < s_m$ on S and any total order on $G \setminus \{e\}$ (no compatibility with the group structure, nor with the action, is needed). Let λ be the edge labeling of $\hat{\mathcal{D}}_n(G, S)$ defined as follows (A, B and C are as in Definition 8.3.1).

$$\lambda(x, y) = \begin{cases} (0, \max(\min A, \min B)) & \text{if } (x, y) \text{ is coherent} \\ (2, \min C, \alpha(x, y)) & \text{if } (x, y) \text{ is non-coherent} \\ (1, k) & \text{if } (x, y) \text{ is colored of color } s_k \\ (1, 2) & \text{if } y = \hat{1}. \end{cases}$$

The values of λ are compared lexicographically. In the second case, $\lambda(x, y)$ is a triple and is lexicographically larger than all the pairs that occur in the other cases.

Lemma 8.3.4 (Non-coherent increasing chains). Let $p_1 \prec p_2 \prec \dots \prec p_k$ be an increasing chain in $\mathcal{D}_n(G, S)$ such that (p_i, p_{i+1}) is non-coherent for all i . Let \tilde{A}, \tilde{B} be non-zero blocks of p_1 such that $A \cup B \subseteq C$ for some non-zero block \tilde{C} of p_k . Then $c(\min A) \neq c(\min B)$.

Proof. Suppose that $\gamma = (p_1 \prec p_2 \prec \cdots \prec p_k)$ is a chain of minimal length for which the lemma is false. By minimality, p_k is the first element of γ where A and B are contained in the same block. In other words, the edge (p_{k-1}, p_k) merges two blocks $A' \supseteq A$ and $B' \supseteq B$ of p_{k-1} into the single block $C = A' \cup B'$ of p_k . Also by minimality, p_1 is the last element of γ where both A and B are not contained in a larger block. Then assume without loss of generality that the edge (p_1, p_2) merges A and some other block of p_1 into a single block $A'' \supseteq A$ of p_2 . Therefore we have the inclusions $A \subsetneq A'' \subseteq A' \subsetneq C$.

Let $\lambda(p_i, p_{i+1}) = (2, j_i, g_i)$. We have the following increasing sequence of labels:

$$(2, j_1, g_1) < (2, j_2, g_2) < \cdots < (2, j_{k-1}, g_{k-1}).$$

Then $j_1 = \min A'' \geq \min C = j_{k-1}$, so $j_1 = j_2 = \cdots = j_{k-1} = \min C$. This means that each edge of γ consists of a merge which involves the element $\min C$. Also, $g_1 < g_2 < \cdots < g_{k-1}$. If we normalize c so that $c(\min C) = e$, by definition of λ we have that:

- $c(\min A) = \begin{cases} e & \text{if } \min A = \min C \\ g_1 & \text{otherwise;} \end{cases}$
- $c(\min B) = \begin{cases} e & \text{if } \min B = \min C \text{ (this can only happen if } k = 2) \\ g_{k-1} & \text{otherwise.} \end{cases}$

If $k = 2$, then $A'' = A \cup B$ and exactly one of $c(\min A)$ and $c(\min B)$ is equal to e . If $k > 2$, we have $c(\min B) = g_{k-1}$, which is different from both e and g_1 . In any case, $c(\min A) \neq c(\min B)$. \square

Theorem 8.3.5 (EL-shellability). The edge labeling λ of Definition 8.3.3 is an EL-labeling of $\hat{\mathcal{D}}_n(G, S)$.

Proof. In order to check Definition 1.5.1, consider an interval $[x, y]$ of $\hat{\mathcal{D}}_n(G, S)$. For $x = y$ or $x \prec y$ there is nothing to prove, so assume $\text{rk}(y) - \text{rk}(x) \geq 2$. Let $x = (\tilde{\beta}, z)$, with underlying partition $\tilde{\beta} = \{\tilde{B}_1, \dots, \tilde{B}_\ell\}$ and zero block Z . Order the blocks of x so that $\min B_1 < \cdots < \min B_\ell$.

Case 1: $y = \hat{1}$. Suppose that $\gamma = (x = p_0 \prec p_1 \prec \cdots \prec p_k \prec \hat{1})$ is an increasing maximal chain in the interval $[x, \hat{1}]$, with $k \geq 1$. Colored edges exist along γ if and only if $S \neq \emptyset$ and $\tilde{\beta} \neq \emptyset$. Since γ is increasing, we can deduce the following: the last edge $(p_k, \hat{1})$ is labeled $(1, 2)$; at most one edge, namely (p_{k-1}, p_k) , is labeled $(1, 1)$ and is colored; all other edges are coherent and their labels are forced to be $(0, \min B_2), (0, \min B_3), \dots, (0, \min B_\ell)$. Notice that $k = \ell$ if (p_{k-1}, p_k) is colored (which happens if and only if $S \neq \emptyset$ and $\tilde{\beta} \neq \emptyset$), otherwise $k = \ell - 1$. In any case, $p_{\ell-1}$ has a single non-zero block $B = B_1 \cup \cdots \cup B_\ell$, with G -coloring b uniquely determined by $b(\min B_1) = \cdots = b(\min B_\ell)$ since all edges from x to $p_{\ell-1}$ are coherent. Therefore $p_{\ell-1}$ is uniquely determined by x . If $k = \ell$ the edge (p_{k-1}, p_k) is colored with color s_1 ,

and thus $p_k = (\emptyset, z_k)$ is uniquely determined by the condition $z_k(\min B) = s_1$. Finally, there is exactly one coherent increasing chain from x to $p_{\ell-1}$, in which (p_i, p_{i+1}) is the coherent edge which merges the blocks $B_1 \cup \dots \cup B_{i+1}$ and B_{i+2} .

The previous argument also shows how to construct an increasing maximal chain $\gamma = (x = p_0 \prec p_1 \prec \dots \prec p_k \prec \hat{1})$ in $[x, \hat{1}]$, so there is exactly one such chain. For all $i \in \{0, \dots, k-1\}$, the label $\lambda(p_i, q)$ is minimized (only) when $q = p_{i+1}$. Therefore γ is lexicographically minimal.

Case 2: $y = (\tilde{\beta}', \emptyset)$, and $\tilde{\beta}'$ has only one non-singleton block \tilde{B} . Let b be the G -coloring of \tilde{B} in y , normalized so that $b(\min B) = e$. Suppose without loss of generality that $B = B_1 \cup \dots \cup B_{\ell'}$, for some $\ell' \leq \ell$. Let $g_i = b(\min B_i)$ for $i = 1, \dots, \ell$. Notice that $g_1 = e$, since $\min B_1 = \min B$.

Suppose that $\gamma = (x = p_0 \prec p_1 \prec \dots \prec p_k = y)$ is an increasing maximal chain in $[x, y]$. Since the zero block of y is empty, along γ there are no colored edges. Then the edges are coherent from x to p_m for some $m \in \{0, \dots, k\}$, and non-coherent from p_m to y . Let $p_m = (\{\tilde{A}_1, \dots, \tilde{A}_s\}, \emptyset)$. Since there is a coherent chain from x to p_m , any block A_j of p_m is a union $B_{i_1} \cup \dots \cup B_{i_r}$ with $g_{i_1} = \dots = g_{i_r}$, and $b(\min A_j) = g_{i_1} = \dots = g_{i_r}$. Lemma 8.3.4 implies that $b(\min A_i) \neq b(\min A_j)$ for all $i \neq j$. Therefore p_m is uniquely determined by x and y .

The part of γ from x to p_m consists only of coherent edges, and it is uniquely determined by x and p_m as in Case 1. Consider now the part of γ from p_m to y . If $p_m = y$ there is nothing to prove, so suppose $p_m \neq y$. The labels from p_m to y take the form

$$(2, j_m, h_m) < (2, j_{m+1}, h_{m+1}) < \dots < (2, j_{k-1}, h_{k-1}).$$

By definition of λ , we have $j_{k-1} = \min B$ and $j_m = \min(A_i \cup A_j)$ for some $i \neq j$. Since $A_i \cup A_j \subseteq B$, we can deduce that $j_m \geq j_{k-1}$. This implies $j_m = j_{m+1} = \dots = j_{k-1} = \min B$. Therefore every non-coherent edge consists of a merge which involves the element $\min B$. Also, we have that $h_m < h_{m+1} < \dots < h_{k-1}$. The elements $e, h_m, h_{m+1}, \dots, h_{k-1}$ of G are all distinct, and by definition of λ they coincide (up to a permutation) with $b(\min A_1), b(\min A_2), \dots, b(\min A_s)$. Then the chain from p_m to y is forced by the order $h_m < h_{m+1} < \dots < h_{k-1}$: first merge the block corresponding to e with the block corresponding to h_m ; then merge the resulting block with the block corresponding to h_{m+1} ; and so on. At each step, the G -coloring is determined by b .

We proved that an increasing chain γ in $[x, y]$ is uniquely determined by x and y , and our argument shows how to construct such a chain. We still need to prove that γ is lexicographically minimal in $[x, y]$. Suppose that a lexicographically minimal chain γ' first differs from γ at some edge (p_r, p'_{r+1}) , i.e. $\lambda(p_r, p'_{r+1}) < \lambda(p_r, p_{r+1})$.

- If $r < m$, the edge (p_r, p_{r+1}) is coherent, so the edge (p_r, p'_{r+1}) must also be coherent. In order to remain in the interval $[x, y]$, a coherent merge between two blocks C_1 and C_2 of p_r is possible only if $b(\min C_1) = b(\min C_2)$. Then $\lambda(p_r, p'_{r+1})$ is minimized for $p'_{r+1} = p_{r+1}$.
- If $r \geq m$, the chain γ' coincides with γ at least up to $p_m = (\{\tilde{A}_1, \dots, \tilde{A}_s\}, \emptyset)$. The edge (p_r, p'_{r+1}) cannot be coherent, because $b(\min A_i) \neq b(\min A_j)$ for all $i \neq j$. Then (p_r, p'_{r+1}) is non-coherent. The second entry of $\lambda(p_r, p'_{r+1})$ is at least $\min B$, so it must be equal to $\min B$ by minimality of γ' . This means that (p_r, p'_{r+1}) consists of a non-coherent merge which involves $\min B$. Then the possible values for the third entry of $\lambda(p_r, p'_{r+1})$ are $\{h_r, h_{r+1}, \dots, h_{k-1}\}$. The smallest one is h_r , which is attained for $p'_{r+1} = p_{r+1}$.

Case 3: $y = (\tilde{\beta}', z')$, and all blocks of $\tilde{\beta}'$ are singletons. Suppose that $\gamma = (x = p_0 \prec p_1 \prec \dots \prec p_k = y)$ is an increasing maximal chain in $[x, y]$. By the structure of y , the last edge (p_{k-1}, y) of γ must be colored. Then the edges along γ are coherent from x to p_m for some $m \in \{0, \dots, k\}$, and colored from p_m to y . Let $p_m = (\{\tilde{A}_1, \dots, \tilde{A}_s\}, z)$. Since all the merges of γ are coherent, if two blocks B_i and B_j of x are contained in the same block A of p_m , then we have $a(\min B_i) = a(\min B_j)$ and therefore $z'(\min B_i) = z'(\min B_j)$. In addition, $z'(\min A_i) \neq z'(\min A_j)$ for all distinct blocks A_i, A_j of p_m , because the colored edges have strictly increasing (and thus distinct) colors. Putting everything together, two blocks B_i and B_j of x are contained in the same block of p_m if and only if $z'(\min B_i) = z'(\min B_j)$. This determines p_m uniquely.

The part of γ from x to p_m is uniquely determined as in Case 1. Then there must be a colored edge for each (non-zero) block of p_m . Their colors are determined, so their order is also determined, because the sequence of colors must be increasing.

Therefore the whole chain γ is uniquely determined by x and y , and once again the previous argument explicitly yields one such chain. Suppose that a lexicographically minimal chain γ' in $[x, y]$ first differs from γ at some edge (p_r, p'_{r+1}) , i.e. $\lambda(p_r, p'_{r+1}) < \lambda(p_r, p_{r+1})$.

- If $r < m$, the edge (p_r, p_{r+1}) is coherent, so the edge (p_r, p'_{r+1}) must also be coherent. In order to remain in the interval $[x, y]$, a coherent merge between two blocks C_1 and C_2 of p_r is possible only if $z'(\min C_1) = z'(\min C_2)$. Then $\lambda(p_r, p'_{r+1})$ is minimized for $p'_{r+1} = p_{r+1}$.
- If $r \geq m$, then the chain γ' coincides with γ at least up to $p_m = (\{\tilde{A}_1, \dots, \tilde{A}_s\}, z'')$. The edge (p_r, p'_{r+1}) cannot be coherent, because $z'(\min A_i) \neq z'(\min A_j)$ for all $i \neq j$. Then (p_r, p'_{r+1}) is colored. The possible colors are given by the values of z' on $\min A_1, \dots, \min A_s$. The smallest color still available is attained for $p'_{r+1} = p_{r+1}$.

Case 4: $y \neq \hat{1}$. Let $y = (\tilde{\beta}', z')$ with underlying partition $\tilde{\beta}' = \{\tilde{B}'_1, \dots, \tilde{B}'_r\}$ and zero block Z' . The interval $[x, y]$ is isomorphic to a product of intervals $[x_i, y_i] \subseteq \mathcal{D}_n(G, S)$ for $0 \leq i \leq r$, where:

- y_0 has the same zero block as y (with the same S -coloring), and all other blocks are singletons;
- for $1 \leq i \leq r$, y_i has exactly one non-singleton block which is equal to \tilde{B}'_i , and an empty zero block.

For each $i \neq j$, the sets of labels used in the intervals $[x_i, y_i]$ and $[x_j, y_j]$ are disjoint. By Case 2 and Case 3, $\lambda|_{[x_i, y_i]}$ is an EL-labeling for all i . Then also $\lambda|_{[x, y]}$ is an EL-labeling by Theorem 1.5.3. \square

Remark 8.3.6. The group structure of G and the G -action on S play a very little role in our EL-labeling. A similar unexpected separation between combinatorics and algebra was already observed in the characteristic polynomial of $\mathcal{D}_n(G, S)$ [BG18, Remark 2.5.3].

8.4 HOMOTOPY TYPE

As in the introduction, let

$$\epsilon = \begin{cases} 0 & \text{if } S \neq \emptyset \\ 1 & \text{if } S = \emptyset. \end{cases} \quad (8.1)$$

In this section we assume not to be in the degenerate case $S = \emptyset$, $G = \{e\}$, and $n = 1$, because $\mathcal{D}_1(\{e\}, \emptyset) \setminus \{\hat{0}\}$ is empty.

Since the poset $\hat{\mathcal{D}}_n(G, S)$ is (EL-)shellable and has rank $n + 1 - \epsilon$, the order complex of $\bar{\mathcal{D}}_n(G, S) = \mathcal{D}_n(G, S) \setminus \{\hat{0}\}$ is homotopy equivalent to a wedge of $(n - 1 - \epsilon)$ -dimensional spheres. The homotopy type is therefore determined by the number of these spheres. As described in Sections 1.4 and 1.5, we have (at least) two ways to determine this number: as an evaluation of the characteristic polynomial, and as the number of decreasing maximal chains in an EL-labeling.

The characteristic polynomial $\chi(t)$ of $\mathcal{D}_n(G, S)$ was computed in [BG18, Theorem 2.5.2]:

$$\chi(t) = \begin{cases} \prod_{i=0}^{n-1} (t - |S| - |G|i) & \text{if } S \neq \emptyset \\ \prod_{i=1}^{n-1} (t - |G|i) & \text{if } S = \emptyset. \end{cases}$$

Therefore the number of spheres is given by

$$(-1)^{\text{rk } \mathcal{D}_n(G, S)} \chi(1) = (-1)^{n-\epsilon} \prod_{i=0}^{n-1} (1 - |S| - |G|i)$$

$$= (-1)^\epsilon \prod_{i=0}^{n-1} (|S| - 1 + |G|i).$$

Notice that the product vanishes for $|S| = 1$. This is correct, because in this case the poset $\bar{\mathcal{D}}_n(G, S)$ is bounded from above, and thus its order complex is contractible.

In the rest of this section we are going to prove the previous formula for the number of spheres by counting the decreasing maximal chains in $\bar{\mathcal{D}}_n(G, S)$, with respect to the EL-labeling of Definition 8.3.3. Some of the arguments below are similar to those of Chapter 7.

Recall from Section 7.6 the definition of increasing ordered trees. It is useful to introduce the following variant of increasing ordered trees, which generalizes the q -blooming trees of Definition 7.6.9.

Definition 8.4.1. Let $q, r \geq 0$ be integers. A (q, r) -blooming tree is an increasing ordered tree with q extra indistinguishable unlabeled nodes appended to the root, and r extra indistinguishable unlabeled nodes appended to each labeled node other than the root. The extra unlabeled nodes are called *blooms*. The only thing that matters about blooms is their position in the total order of the children of a node.

Blooms can be regarded as separators placed in the list of the children of a node. See Figure 8.1 for a few examples.

Lemma 8.4.2. The number of (q, r) -blooming trees on $n + 1$ (labeled) nodes is

$$\prod_{i=0}^{n-1} (q + 1 + (r + 2)i).$$

Proof. The proof is by induction on n . For $n = 0$, there is only one (q, r) -blooming tree, which consists of the root with q blooms attached to it. Let T be any (q, r) -blooming tree on n nodes with $n \geq 1$. The tree T has $q + (n - 1)r$ blooms, so there are exactly $2n - 1 + q + (n - 1)r$ positions where an additional node with label n can be attached (together with its r new blooms) in order to obtain a (q, r) -blooming tree on $n + 1$ nodes. Every (q, r) -blooming tree on $n + 1$ nodes is obtained exactly once in this way. \square

Theorem 8.4.3 (Homotopy type of $\bar{\mathcal{D}}_n(G, S)$). The order complex of the poset $\bar{\mathcal{D}}_n(G, S)$ is homotopy equivalent to a wedge of

$$(-1)^\epsilon \prod_{i=0}^{n-1} (|S| - 1 + |G|i)$$

$(n - 1 - \epsilon)$ -dimensional spheres (where ϵ is defined in (8.1)), except for the empty poset $\bar{\mathcal{D}}_1(\{e\}, \emptyset)$.

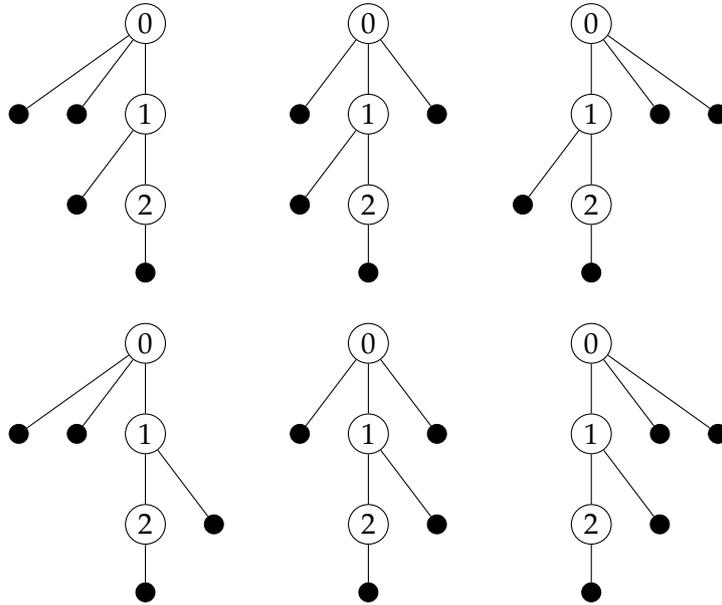


Figure 8.1: All different $(2, 1)$ -blooming trees constructed from the leftmost increasing ordered tree of Figure 7.1. These are 6 of the 18 different $(2, 1)$ -blooming trees on 3 (labeled) nodes. Blooms are shown as smaller black (unlabeled) nodes.

Proof. We want to compute the cardinality of the set D of the decreasing maximal chains from $\hat{0}$ to $\hat{1}$ in $\hat{\mathcal{D}}_n(G, S)$. A chain $\gamma \in D$ consists of:

- a sequence of non-coherent edges, labeled $(2, *, *)$;
- then, a sequence of colored edges, labeled $(1, *)$ with $* \geq 2$;
- finally, one edge labeled $(1, 2)$.

Case 1: $|G| = 1$. In this case, non-coherent edges do not exist. Then a decreasing chain $\gamma \in D$ is determined by a permutation of $[n]$ (which encodes the order in which the elements of $[n]$ are colored) and a decreasing sequence of n labels

$$(1, |S|) \geq (1, h_1) \geq (1, h_2) \geq \dots \geq (1, h_n) \geq (1, 2)$$

(which are to be assigned to the colored edges of γ). The number of decreasing chains in $\mathcal{D}_n(\{e\}, S)$ is therefore

$$n! \cdot \binom{n + |S| - 2}{n} = \frac{(n + |S| - 2)!}{(|S| - 2)!}$$

if $|S| \geq 2$, and 0 if $|S| \leq 1$. From now on, assume $|G| \geq 2$.

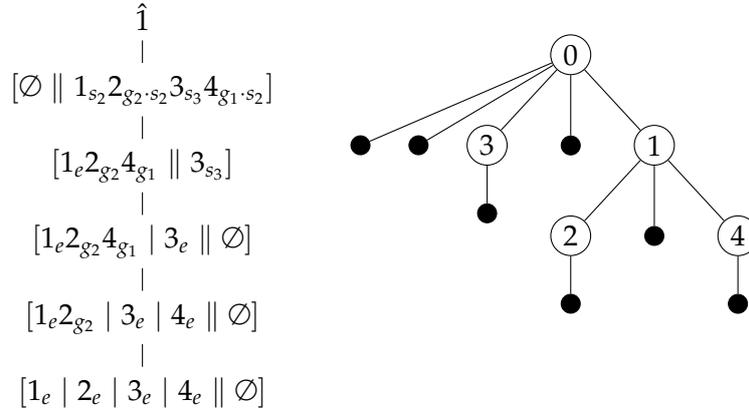


Figure 8.2: A decreasing chain (for $n = 4$, $|G| = 3$, and $|S| = 5$), and the corresponding $(3, 1)$ -blooming tree.

Case 2: $|S| = 1$. Every chain contains at least one colored edge which is labeled $(1, 1)$. Thus there are no decreasing chains.

Case 3: $|G| \geq 2$ and $|S| \geq 2$. We are going to construct a bijection ψ between D and the set \mathcal{T} of $(|S| - 2, |G| - 2)$ -blooming trees on $n + 1$ nodes. Let $S = \{s_1 < s_2 < \dots < s_m\}$ and $G \setminus \{e\} = \{g_1 < g_2 < \dots < g_k\}$. The idea is: for every non-coherent merge of blocks A and B , there is an edge between $\min A$ and $\min B$; for every S -coloring of a block B , there is an edge between 0 and $\min B$; the blooms indicate when to stop using a certain s_i (or g_i) and start using s_{i-1} (or g_{i-1}).

The bijection $\psi: D \rightarrow \mathcal{T}$ is defined as follows. Let $\gamma \in D$ be a decreasing chain. In order to construct the tree $\psi(\gamma) \in \mathcal{T}$, start with a disconnected graph on $n + 1$ vertices labeled $0, 1, \dots, n$. Every time we say “attach the node u to the node v ” we mean “create an edge between v and u , so that u becomes the *last* child of v in the total order of the children of v .”

First examine the colored edges along γ , in order. For each colored edge (x, y) , which colors a block B and has label $\lambda(x, y) = (1, i)$, do the following.

- (1) If the root (i.e. the node 0) has less than $m - i$ blooms, attach a new bloom to it; repeat this step until the root has exactly $m - i$ blooms.
- (2) Attach the node $\min B$ to the root.

Notice that, by monotonicity of the labels along γ , the number of blooms required in step (1) is weakly increasing and it varies between 0 and $m - 2$.

Then examine the non-coherent edges along γ , in order. For each non-coherent edge (x, y) , which merges blocks A and B with $\min A < \min B$ and has label $\lambda(x, y) = (2, \min A, g_i)$, do the following.

- (1) If the node $\min A$ has less than $k - i$ blooms, attach a new bloom to it; repeat this step until the node $\min A$ has exactly $k - i$ blooms.
- (2) Attach the node $\min B$ to the node $\min A$.

Again, by monotonicity of the labels along γ , the number of blooms required in step (1) is weakly increasing and it varies between 0 and $k - 1 = |G| - 2$.

Attach new blooms to the tree, so that the root has $m - 2$ blooms and every other labeled node has $k - 1$ blooms. Define $\psi(\gamma)$ as the tree resulting from this construction. See Figure 8.2 for an example.

To prove that ψ is a bijection, we explicitly define its inverse $\psi^{-1}: \mathcal{T} \rightarrow D$. Let $T \in \mathcal{T}$ be a tree. Start with $\gamma = (\hat{0})$ (a chain with one element). Consider the set of couples

$$\{(u, v) \mid u, v \text{ are labeled nodes of } T, \text{ and } v \text{ is a child of } u\},$$

totally ordered by: $(u_1, v_1) < (u_2, v_2)$ if $u_1 > u_2$, or $u_1 = u_2$ and v_1 comes before v_2 in the total order of the children of $u_1 = u_2$. Each of the nodes $1, \dots, n$ appears exactly once as the second entry of a couple (u, v) . With (u, v) running through this ordered set of couples, do the following.

- Let i be the number of blooms attached to u which come before v in the total order of the children of u .
- If $x \in \mathcal{D}_n(G, S)$ is the last element of γ , construct $y \succ x$ as follows.
 - Case $u > 0$. Merge the block containing u and the block containing v so that $\lambda(x, y) = (2, u, g_{k-i})$.
 - Case $u = 0$. Color the block containing v so that $\lambda(x, y) = (1, m - i)$.

In both cases, the element $y \succ x$ is uniquely determined by the given conditions.

- Extend γ by adding y after x .

Extend γ once more, by adding $\hat{1}$ as the last element. Then $\psi^{-1}(T) = \gamma$.

By Lemma 8.4.2, the number of $(|S| - 2, |G| - 2)$ -blooming trees on $n + 1$ nodes is

$$\prod_{i=0}^{n-1} (|S| - 1 + |G|i).$$

This is also the cardinality of D .

Case 4: $S = \emptyset$. Differently from before, in this case there are no colored edges, and the zero blocks are always empty. Then D is in bijection with the set of $(|G| - 2, |G| - 2)$ -blooming trees on n vertices labeled $1, 2, \dots, n$. The bijection is constructed as in the case $|S| \geq 2$, except that there is no special “node 0” anymore. By Lemma 8.4.2, the number of such trees is

$$\prod_{i=0}^{n-2} (|G| - 1 + |G|i) = \prod_{i=1}^{n-1} (-1 + |G|i) = - \prod_{i=0}^{n-1} (-1 + |G|i).$$

This is also the cardinality of D . \square

In the case of Dowling lattices $\mathcal{D}_n(G)$, obtained by setting $|S| = 1$, Theorem 8.4.3 is trivial because $\mathcal{D}_n(G)$ contains a top element (\emptyset, \hat{z}) . The subposet $\tilde{\mathcal{D}}_n(G) = \mathcal{D}_n(G) \setminus \{\hat{0}, (\emptyset, \hat{z})\}$ is shellable, and its order complex is homotopy equivalent to a wedge of spheres in bijection with the decreasing chains from $\hat{0}$ to (\emptyset, \hat{z}) . By an argument similar to that of Theorem 8.4.3, these decreasing chains are counted by $(0, |G| - 2)$ -blooming trees on $n + 1$ vertices. By Lemma 8.4.2, their number is

$$\prod_{i=0}^{n-1} (1 + |G|i).$$

A similar description of the generators of the homology of Dowling lattices was given in [GW00, Section 4], in terms of labeled forests. The formula for the number of generators was first found in [Dow73].

8.5 SHELLABILITY OF SUBPOSETS

We now turn our attention to the subposet $\mathcal{P}_n(G, S, T)$ of $\mathcal{D}_n(G, S)$ introduced in Section 8.2, where T is a G -invariant subset of S . Throughout this section, assume $n \geq 2$ — because $\mathcal{P}_1(G, S, T) = \mathcal{D}_1(G, T)$ — and $|S| \geq 1$. Then $\text{rk } \mathcal{P}_n(G, S, T) = n$. Also, let $\tilde{\mathcal{P}}_n(G, S, T) = \mathcal{P}_n(G, S, T) \setminus \{\hat{0}\}$.

We are going to prove that $\mathcal{P}_n(G, S, T)$ is shellable if G acts trivially on $S \setminus T$. In general, however, $\mathcal{P}_n(G, S, T)$ is not shellable. In the first part of this section we construct a wide family of non-shellable examples (Proposition 8.5.6), whereas in the second part we prove shellability if the G -action on $S \setminus T$ is trivial (Theorem 8.5.8). Let us start with two simple examples.

Example 8.5.1. Let $G = \mathbb{Z}_2 = \{e, g\}$ act non-trivially on $S = \{+, -\}$, as in [BG18, Example 2.2.3]. The Hasse diagram of $\mathcal{P}_2(G, S, \emptyset)$ is shown in Figure 8.3. The order complex of $\tilde{\mathcal{P}}_2(G, S, \emptyset)$ is 1-dimensional and disconnected, therefore it is not shellable. This example is a special case of Proposition 8.5.6 below. In view of Theorem 8.5.8, this is the smallest non-shellable example.

Example 8.5.2. Let $G = \mathbb{Z}_4 = \{e, g, g^2, g^3\}$ act non-trivially on $S = \{+, -\}$. The Hasse diagram of $\mathcal{P}_2(G, S, \emptyset)$ is shown in Figure 8.4. As in Example 8.5.1, the order complex Δ of $\tilde{\mathcal{P}}_2(G, S, \emptyset)$ is 1-dimensional and disconnected, therefore it is not shellable. In this example, $\Delta \simeq S^1 \sqcup S^1$ is not even a wedge of spheres.

We are going to prove a “reduction lemma” that allows to construct a wide family of non-shellable examples. At the same time, it gives interesting homotopy equivalences between subposets of different S -Dowling posets.

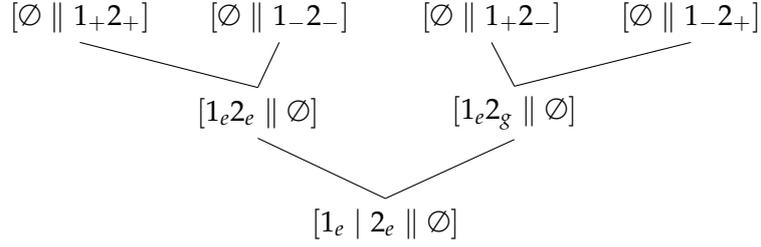


Figure 8.3: The Hasse diagram of $\mathcal{P}_2(\mathbb{Z}_2, \{+, -\}, \emptyset)$ where \mathbb{Z}_2 acts non-trivially on $\{+, -\}$ (see Example 8.5.1).

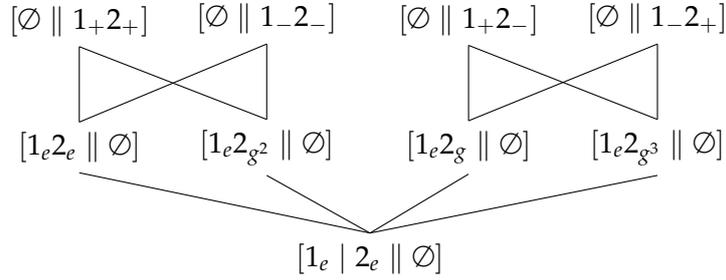


Figure 8.4: The Hasse diagram of $\mathcal{P}_2(\mathbb{Z}_4, \{+, -\}, \emptyset)$ where \mathbb{Z}_4 acts non-trivially on $\{+, -\}$ (see Example 8.5.2).

Lemma 8.5.3 (Orbit reduction). Suppose that $O \subseteq S \setminus T$ is a G -orbit with a trivial stabilizer, i.e. $|O| = |G|$. Then there is a homotopy equivalence $\bar{\mathcal{P}}_n(G, S, T) \simeq \bar{\mathcal{P}}_n(G, S \setminus O, T)$.

Proof. Fix an element $\bar{s} \in O$. Since O has a trivial stabilizer, this choice induces a bijection $\varphi: G \xrightarrow{\cong} O$ given by $g \mapsto g \cdot \bar{s}$. We are going to construct a (descending) closure operator $f: \mathcal{P}_n(G, S, T) \rightarrow \mathcal{P}_n(G, S \setminus O, T) \subseteq \mathcal{P}_n(G, S, T)$, i.e. an order-preserving map satisfying $f(x) \leq x$ for all $x \in \mathcal{P}_n(G, S, T)$.

Let $x = (\tilde{\beta}, z) \in \mathcal{P}_n(G, S, T)$, and let Z be the zero block of x . Consider the subset $Z_O = \{i \in Z \mid z(i) \in O\}$ of Z consisting of the elements colored by the orbit O . To define $f(x)$, remove Z_O from the zero block and add $B = Z_O$ as a non-zero block with G -coloring $b = \varphi^{-1} \circ z|_{Z_O}$:

$$f(x) = (\tilde{\beta} \cup \{B\}, z|_{Z \setminus Z_O}).$$

Notice that f does not depend on the initial choice of $\bar{s} \in O$. Indeed, different choices of \bar{s} yield \sim -equivalent G -colorings of B .

Clearly $f(x) \leq x$, because either $f(x) = x$ (if $Z_O = \emptyset$) or x can be obtained from $f(x)$ by coloring the block B . Also, f is order-preserving:

- if $(x, y) \in \mathcal{E}(\mathcal{P}_n(G, S, T))$ is an edge of type “merge”, or is an edge of type “color” which uses colors in $S \setminus O$, then $(f(x), f(y))$ is an edge of the same type;
- if $(x, y) \in \mathcal{E}(\mathcal{P}_n(G, S, T))$ is an edge of type “color” which uses colors in O , then either $x = f(x) = f(y)$ (if $Z_O = \emptyset$ in x) or $(f(x), f(y))$ is an edge of type “merge”.

In addition, f is the identity on $\mathcal{P}_n(G, S \setminus O, T)$, and so f is surjective.

In the definition of $f(x)$, either $f(x) = x$ (if $Z_O = \emptyset$) or $f(x)$ contains the block B which has at least 2 elements (because $O \subseteq S \setminus T$; see the definition of the subposet $\mathcal{P}_n(G, S, T)$). Therefore $f^{-1}(\hat{0}) = \{\hat{0}\}$. Then f restricts to a surjective closure operator $\bar{f}: \bar{\mathcal{P}}_n(G, S, T) \rightarrow \bar{\mathcal{P}}_n(G, S \setminus O, T)$. By [Bjö95, Corollary 10.12], \bar{f} induces a homotopy equivalence between the associated order complexes. \square

Remark 8.5.4. The closure operator f of the previous proof also satisfies $f^2 = f$. Then, by [Koz06, Theorem 2.1], we actually have that the order complex of $\bar{\mathcal{P}}_n(G, S, T)$ collapses onto the order complex of $\bar{\mathcal{P}}_n(G, S \setminus O, T)$.

Remark 8.5.5. The combinatorics of $\bar{\mathcal{P}}_n(G, S, T)$ is related to the topology of the complement of an arrangement $\mathcal{A}_n(G, X)$ of singular subspaces in X^n . In this sense an orbit O as in the statement of Lemma 8.5.3 is “redundant”, as it consists of non-singular points. This provides a topological interpretation of Lemma 8.5.3.

Proposition 8.5.6. Let $n \geq 2$ and $|S| \geq 1$. Suppose that all the G -orbits in $S \setminus T$ have a trivial stabilizer. Then $\bar{\mathcal{P}}_n(G, S, T)$ is homotopy equivalent to a wedge of d -dimensional spheres with

$$d = \begin{cases} n - 1 & \text{if } T \neq \emptyset \\ n - 2 & \text{if } T = \emptyset. \end{cases}$$

In particular, if $|G| \geq 2$ and all the G -orbits in S have a trivial stabilizer, the poset $\mathcal{P}_n(G, S, \emptyset)$ is not shellable.

Proof. For the first part, a repeated application of Lemma 8.5.3 yields $\bar{\mathcal{P}}_n(G, S, T) \simeq \bar{\mathcal{P}}_n(G, T, T) = \bar{\mathcal{D}}_n(G, T)$. Then the homotopy type of $\bar{\mathcal{D}}_n(G, T)$ is given by Theorem 8.4.3. In particular, the dimension d of the spheres equals $\text{rk } \mathcal{D}_n(G, T) - 1$, which is $n - 1$ for $T \neq \emptyset$ and $n - 2$ for $T = \emptyset$.

For the second part, $\bar{\mathcal{P}}_n(G, S, \emptyset) \simeq \bar{\mathcal{D}}_n(G, \emptyset)$. Since $|S| \geq 1$, the poset $\mathcal{P}_n(G, S, \emptyset)$ has rank n . For $\mathcal{P}_n(G, S, \emptyset)$ to be shellable, the order complex of $\bar{\mathcal{P}}_n(G, S, \emptyset)$ must be homotopy equivalent to a wedge of $(n - 1)$ -dimensional spheres. The hypothesis $|G| \geq 2$ ensures that $\bar{\mathcal{P}}_n(G, S, \emptyset) \simeq \bar{\mathcal{D}}_n(G, \emptyset)$ is a wedge of a positive number of $(n - 2)$ -dimensional spheres (by Theorem 8.4.3). Then $\mathcal{P}_n(G, S, \emptyset)$ is not shellable. \square

The second part of this proposition yields a large family of examples of non-shellable subposets of S -Dowling posets, generalizing Example 8.5.1. At the same time it shows that this family is still well-behaved, as $\tilde{\mathcal{P}}_n(G, S, \emptyset)$ is homotopy equivalent to a wedge of spheres.

We now prove that $\mathcal{P}_n(G, S, T)$ is shellable if the G -action on $S \setminus T$ is trivial. This generalizes Theorem 7.5.1 (EL-shellability of arrangements defined by root systems). Let $\hat{\mathcal{P}}_n(G, S, T) = \mathcal{P}_n(G, S, T) \cup \{\hat{1}\} \subseteq \hat{\mathcal{D}}_n(G, S)$. For an element $x = (\tilde{\beta}, z) \in \mathcal{P}_n(G, S, T)$, define $S(x) \subseteq S$ as the image of the coloring map $z: Z \rightarrow S$. Consider the following edge labeling, which is a slightly modified version of the edge labeling of Definition 8.3.3 (see also Definition 7.4.1).

Definition 8.5.7 (Edge labeling of $\hat{\mathcal{P}}_n(G, S, T)$). Fix arbitrary total orders on S and on $G \setminus \{e\}$. For a subset $R \subseteq S$, let $R_{\leq s} = \{r \in R \mid r \leq s\}$. Let μ be the edge labeling of $\hat{\mathcal{P}}_n(G, S, T)$ defined as follows (A, B and C are as in Definition 8.3.1).

$$\mu(x, y) = \begin{cases} (0, \max(\min A, \min B)) & \text{if } (x, y) \text{ is coherent} \\ (2, \min C, \alpha(x, y)) & \text{if } (x, y) \text{ is non-coherent} \\ (1, |S(x)_{\leq s}|) & \text{if } (x, y) \text{ is colored of a color } s \in S(x) \\ (1, |S_{\leq s} \cup S(x)|) & \text{if } (x, y) \text{ is colored of a color } s \notin S(x) \\ (1, 2) & \text{if } y = \hat{1}. \end{cases}$$

The difference with the edge labeling λ of Definition 8.3.3 is in the labels of colored edges: λ only depends on the color s , whereas μ favors colors which already belong to $S(x)$.

Theorem 8.5.8 (EL-shellability of subposets). Let $n \geq 2$, and suppose that the G -action on $S \setminus T$ is trivial. Then the edge labeling μ of Definition 8.5.7 is an EL-labeling of $\hat{\mathcal{P}}_n(G, S, T)$.

Proof. Since the edge labelings λ and μ almost coincide, most of the proof of Theorem 8.3.5 also applies here. We refer to that proof (with its notation), and we only highlight the differences. Let $[x, y]$ be an interval in $\hat{\mathcal{P}}_n(G, S, T)$.

Case 1: $y = \hat{1}$. The only difference is that, if $k = \ell$ and $Z \neq \emptyset$, the edge (p_{k-1}, p_k) is colored with color $\min S(x)$ (and not with color s_1). This modification assures that p_k belongs to the subposet $\mathcal{P}_n(G, S, T)$: if $|Z| \neq \emptyset$, the color of the edge (p_{k-1}, p_k) was already used in x ; if $|Z| = \emptyset$, the edge (p_{k-1}, p_k) colors $n \geq 2$ elements at the same time.

Case 2: $y = (\tilde{\beta}', \emptyset)$, and $\tilde{\beta}'$ has only one non-singleton block \tilde{B} . In this case the edge labelings λ and μ coincide, and $[x, y]$ is also an interval of $\mathcal{D}_n(G, S)$. Therefore the proof works without changes.

Case 3: $y = (\tilde{\beta}', z')$, and all blocks of $\tilde{\beta}'$ are singletons. Here we only have to show that the increasing chain $\gamma = (x = p_0 \prec p_1 \prec \cdots \prec p_k = y)$ is

contained in the subposet $\mathcal{P}_n(G, S, T)$. Until the element $p_m \in \gamma$, only coherent edges are used, so $p_i \in \mathcal{P}_n(G, S, T)$ for $i \leq m$. Suppose that $p_i \in \mathcal{P}_n(G, S, T)$ for some $i \in \{m, m+1, \dots, k-1\}$. We want to prove that $p_{i+1} = (\tilde{\beta}'', z'')$ also belongs to $\mathcal{P}_n(G, S, T)$. The edge (p_i, p_{i+1}) is colored of some color s . If $s \in T$ then $p_{i+1} \in \mathcal{P}_n(G, S, T)$ because no color of $S \setminus T$ is added. If $s \notin T$, the G -action is trivial on s . Also, (p_i, p_{i+1}) is the only edge of γ with color s . Therefore every other colored edge of γ has a color $s' \neq s$, and cannot create elements colored by s (because s and s' are in different orbits). Then $z'^{-1}(s) = z''^{-1}(s)$, and so $|z''^{-1}(s)| = |z'^{-1}(s)| \neq 1$ because $y \in \mathcal{P}_n(G, S, T)$. Therefore $p_{i+1} \in \mathcal{P}_n(G, S, T)$. By induction, the entire chain γ is contained in $\mathcal{P}_n(G, S, T)$.

Case 4: $y \neq \hat{1}$. The proof works without changes in this case. \square

Remark 8.5.9. For $T = S$, Theorem 8.5.8 says that μ is an EL-labeling of $\hat{\mathcal{D}}_n(G, S)$.

Remark 8.5.10. Let $G = \mathbb{Z}_2$ act on $S = \{+, -, 0\}$ by exchanging $+$ and $-$. A computer check shows that the order complex of the poset $\hat{P}_3(G, S, \emptyset)$ is shellable. However, the edge labeling of Definition 8.5.7 is not an EL-labeling of $\hat{P}_3(G, S, \emptyset)$.

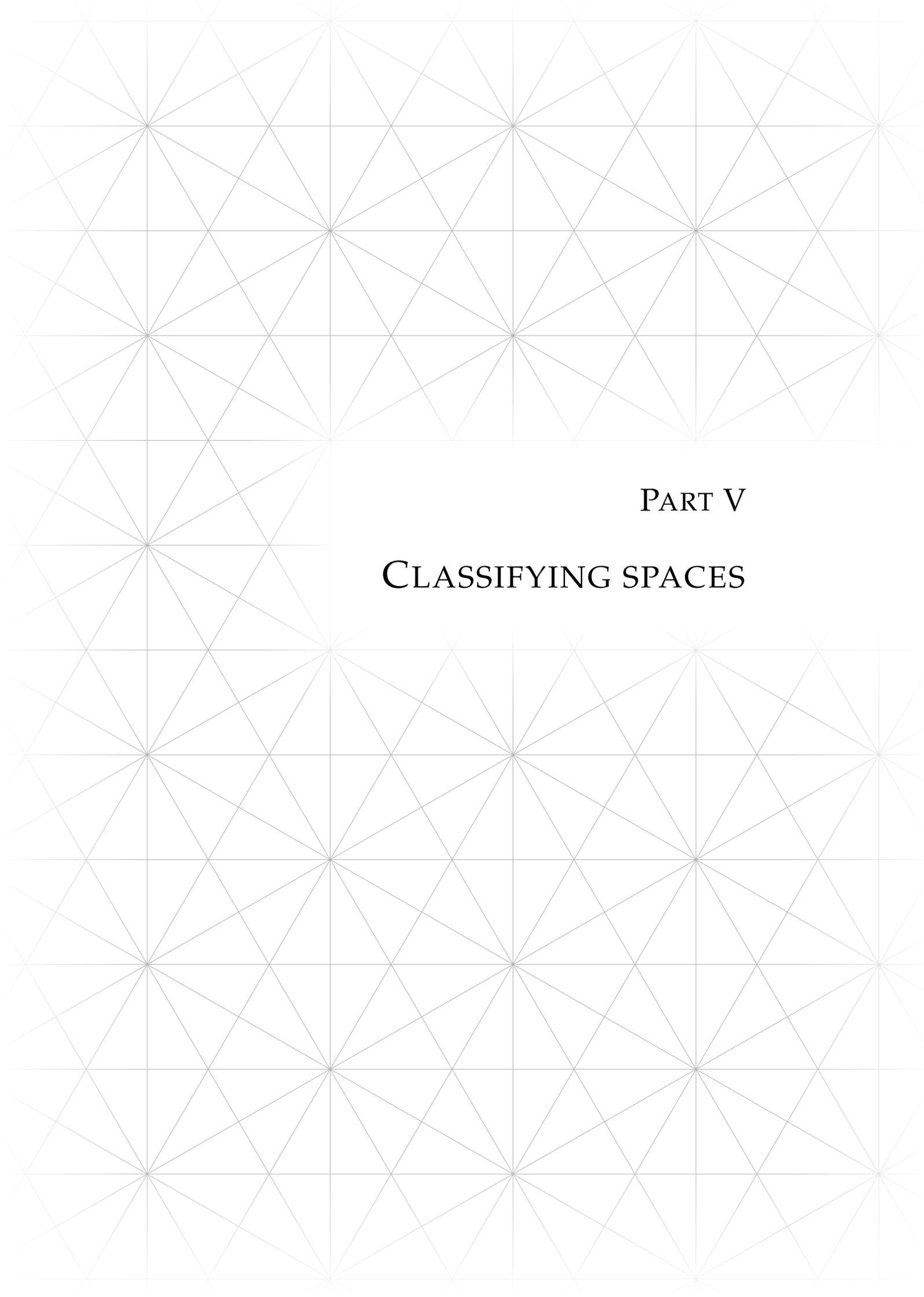
It seems difficult in general to derive an explicit formula for the number of decreasing maximal chains in $\hat{\mathcal{P}}_n(G, S, T)$, with respect to the edge labeling of Definition 8.5.7. This was done in Section 7.6 for posets of layers of arrangements defined by root systems (i.e. with $G = \mathbb{Z}_2$ acting trivially on S , and $|T| = 0, 1$).

Lemma 8.5.3 and Theorem 8.5.8 yield the following strengthening of Proposition 8.5.6.

Theorem 8.5.11. Let $n \geq 2$ and $|S| \geq 1$. Suppose that all the G -orbits in $S \setminus T$ either have cardinality 1, or have a trivial stabilizer. Then $\bar{\mathcal{P}}_n(G, S, T)$ is homotopy equivalent to a wedge of d -dimensional spheres with

$$d = \begin{cases} n-1 & \text{if } T \neq \emptyset \text{ or at least one } G\text{-orbit is trivial} \\ n-2 & \text{otherwise.} \end{cases}$$

Proof. Remove all orbits of $S \setminus T$ with a trivial stabilizer, by a repeated application of Lemma 8.5.3. The remaining poset is shellable by Theorem 8.5.8, and thus it is homotopy equivalent to a wedge of spheres. \square



PART V

CLASSIFYING SPACES

Finite classifying spaces for affine Artin groups

9.1 INTRODUCTION

The $K(\pi, 1)$ conjecture is one of the most important open problems in the theory of Artin groups. It states that the orbit configuration space Y_W associated with a Coxeter group W is a classifying space for the corresponding Artin group G_W (see Section 2.6). It implies that all Artin groups admit a finite classifying space. So far, the $K(\pi, 1)$ conjecture has been proved for all the spherical groups, for some affine groups (\tilde{A}_n , \tilde{B}_n , \tilde{C}_n , and \tilde{G}_2), and for a few other families of Artin groups. In particular, it is still unsolved for the affine groups of type \tilde{D}_n , and for the exceptional affine groups of dimension greater than 2 (see Figures 2.7 and 2.8).

In a recent work [MS17], McCammond and Sulway described how to realize affine Artin groups as subgroups of certain *braided crystallographic groups*, which admit a Garside structure. A group with a Garside structure has many interesting properties. In particular, it has a solvable word problem and it admits a finite-dimensional classifying space (see [BKL98, DP99, Bra01, BW02, Deh02a, Deh02b, Bes03, CMW04, McC05, DDM13]). Using Garside theory, in [MS17] it was proved that affine Artin groups are torsion-free centerless groups, and have a solvable word problem and a finite-dimensional classifying space. However, this classifying space is obtained as an infinite cover and does not have an explicit description. In [MS17, Conjecture 11.10], the authors asked if their classifying space is homotopy equivalent to the orbit configuration space Y_W (see Section 2.6), thus establishing the $K(\pi, 1)$ conjecture.

In this chapter we show how to construct finite classifying spaces for all affine Artin groups. Our construction is based on the work of McCam-

mond and Sulway [MS17], as well as on some related papers [BM15, McC15]. Therefore we establish the following result.

Theorem 9.7.7. Every affine Artin group admits a finite classifying space.

We also prove the same property for almost all of the braided crystallographic groups introduced in [MS17].

Theorem 9.5.12. If W is an affine Coxeter group not of type \tilde{A}_n , the braided crystallographic group associated to W admits a finite classifying space.

In order to relate the classifying space of Theorem 9.7.7 (which will be denoted by K'_W) to the orbit configuration space Y_W , we introduce the *dual Salvetti complex* X'_W . This complex naturally sits inside K'_W , and it is homotopy equivalent to Y_W thanks to the following result.

Theorem 9.4.3. The dual Salvetti complex X'_W is homotopy equivalent to the Salvetti complex X_W .

As a consequence, in order to prove the $K(\pi, 1)$ conjecture, it is enough to show that K'_W collapses onto X'_W . We show this in the case \tilde{D}_4 .

Theorem 9.8.1. The $K(\pi, 1)$ conjecture holds for the Artin group of type \tilde{D}_4 .

This method to prove the $K(\pi, 1)$ conjecture seems to be promising, and we hope to extend it to all affine Artin groups.

The rest of this chapter is structured as follows. Sections 9.2 and 9.3 contain background definitions and constructions, especially from [MS17]. In Section 9.4 we introduce the dual Salvetti complex, and we prove that it is homotopy equivalent to the usual Salvetti complex. In Section 9.5 we show how to construct finite classifying spaces for braided crystallographic groups. In Section 9.6 we recall from [MS17] the definition and some properties of horizontal and diagonal groups, and we prove a few propositions that are needed later. In Section 9.7 we construct finite classifying spaces for all affine Artin groups. Finally, in Section 9.8 we specialize our constructions to \tilde{D}_4 , and we prove the $K(\pi, 1)$ conjecture in this case.

9.2 INTERVAL GROUPS

In this section we recall the construction of interval groups and how they give rise to Garside structures. We summarize the exposition of [MS17, Section 2], and refer to [Bes03, Dig06, Dig12, DDM13, DDG⁺15, McC15, MS17] for a complete reference.

Let G be a group, with a (possibly infinite) generating set $R \subseteq G$ such that $R = R^{-1}$. Suppose that the elements of R are assigned positive weights bounded away from 0 that form a discrete subset of the positive real numbers. Assume that r and r^{-1} have the same weight, for every $r \in R$. For every $x \in G$, denote by $l(x)$ the minimum sum of the weights of some generators

$r_1, r_2, \dots, r_k \in R$ such that $r_1 r_2 \cdots r_k = x$. In other words, $l(x)$ is the distance between 1 and x in the weighted right Cayley graph of G (with respect to the weighted generating set R).

The group G becomes a poset if we set $x \leq y$ whenever $l(x) + l(x^{-1}y) = l(y)$, i.e. if there is a minimal length factorization of y that starts with a minimal length factorization of x . Given an element $g \in G$, denote by $[1, g]^G \subseteq G$ the interval between 1 and g (with respect to the partial order \leq in G). The Hasse diagram of $[1, g]^G$ embeds into the Cayley graph of G . Every edge of the Hasse diagram is of the form (x, xr) for some $r \in R$, and we label it by r .

Definition 9.2.1 (Interval group [MS17, Definition 2.6]). The *interval group* G_g is the group presented as follows. Let R_0 be the subset of R consisting of the labels of edges in $[1, g]^G$. The group G_g has R_0 as its generating set, and relations given by all the closed loops inside the Hasse diagram of $[1, g]^G$.

The interval $[1, g]^G$ is *balanced* if the following condition is satisfied: for every $x \in G$, we have $l(x) + l(x^{-1}g) = l(g)$ if and only if $l(gx^{-1}) + l(x) = l(g)$. This condition is automatically satisfied if the generating set R is closed under conjugation and the weight of a generator is equal to the weight of all its conjugates.

Theorem 9.2.2 ([Bes03, Theorem 0.5.2], [MS17, Proposition 2.11]). If the interval $[1, g]^G$ is a balanced lattice, then the group G_g is a Garside group.

The main reason we are interested in Garside groups is because their classifying spaces can be constructed explicitly.

Definition 9.2.3 (Interval complex). Realize the standard d -simplex Δ^d as the set of points $(a_1, a_2, \dots, a_d) \in \mathbb{R}^d$ such that $1 \geq a_1 \geq a_2 \geq \dots \geq a_d \geq 0$. The *interval complex* associated to the interval $[1, g]^G$ is a Δ -complex (in the sense of [Hat02]) having a d -simplex $[x_1 | x_2 | \dots | x_d]$ for every $x_1, x_2, \dots, x_d \in [1, g]^G$ such that:

- (i) $x_i \neq 1$ for all i ;
- (ii) $x_1 x_2 \cdots x_d \in [1, g]^G$;
- (iii) $l(x_1 x_2 \cdots x_d) = l(x_1) + l(x_2) + \dots + l(x_d)$.

The faces of a simplex $[x_1 | \dots | x_d]$ are as follows.

- The face $\{1 = a_1 \geq a_2 \geq \dots \geq a_d \geq 0\}$ of $[x_1 | \dots | x_d]$ is glued to the $(d-1)$ -simplex $[x_2 | \dots | x_d]$ by sending $(1, a_2, \dots, a_d)$ to $(a_2, \dots, a_d) \in \Delta^{d-1}$.
- For $1 \leq i \leq d-1$, the face $\{1 \geq a_1 \geq \dots \geq a_i = a_{i+1} \geq \dots \geq a_d \geq 0\}$ of $[x_1 | \dots | x_d]$ is glued to the $(d-1)$ -simplex $[x_1 | \dots | x_i x_{i+1} | \dots | x_d]$ by sending $(a_1, \dots, a_i, a_i, a_{i+2}, \dots, a_d)$ to $(a_1, \dots, a_i, a_{i+2}, \dots, a_d) \in \Delta^{d-1}$.

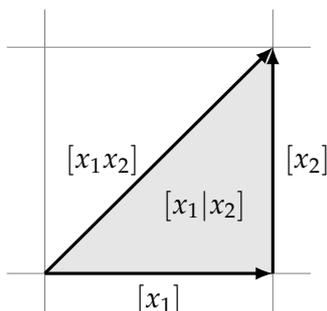


Figure 9.1: Realization in \mathbb{R}^2 of a 2-simplex $[x_1|x_2]$.

- Finally, the face $\{1 \geq a_1 \geq \dots \geq a_d = 0\}$ of $[x_1|\dots|x_d]$ is glued to the $(d-1)$ -simplex $[x_1|\dots|x_{d-1}]$ by sending $(a_1, \dots, a_{d-1}, 0)$ to $(a_1, \dots, a_{d-1}) \in \Delta^{d-1}$.

Notice that there is a unique vertex, which is indicated by $[\]$. See Figure 9.1 for an example.

The fundamental group of the interval complex associated to $[1, g]^G$ is G_g . This can be easily checked by looking at the 2-skeleton. If $[1, g]^G$ is a lattice, then G_g is a Garside group, and the interval complex of $[1, g]^G$ is a $K(G_g, 1)$ (see for example [McC05, Definition 1.6]). However, interval complexes will be important also in non-Garside cases.

9.3 DUAL ARTIN GROUPS AND BRAIDED CRYSTALLOGRAPHIC GROUPS

In this section we recall some constructions and results from [McC15, MS17]. Let (W, S) be a Coxeter system, and let R be the set of all reflections of W . Assign weight 1 to every reflection $r \in R$. For any fixed total order of the elements of S , the product of these generators in this order is a *Coxeter element*. For every Coxeter element w there is a dual Artin group.

Definition 9.3.1 ([BW02, Bes03]). The *dual Artin group with respect to w* is the interval group W_w constructed using R as the generating set of W .

The generating set $R_0 \subseteq R$ of a dual Artin group contains S . Then there is a natural group homomorphism from the usual Artin group G_W to the dual Artin group W_w .

Theorem 9.3.2 ([BW02, Bes03, MS17]). For all finite and affine Coxeter groups W , the natural homomorphism $G_W \rightarrow W_w$ is an isomorphism.

It is not known in general if a dual Artin group is isomorphic to the corresponding Artin group.

The main reason to introduce dual Artin groups is to apply Theorem 9.2.2 and obtain a Garside structure. Everything works smoothly if W is finite.

Theorem 9.3.3 ([BW08]). If W is a finite Coxeter group, the interval $[1, w]^W$ is a lattice for every Coxeter element w . Therefore the dual Artin group W_w is a Garside group.

Things get more complicated for affine Coxeter groups. Notice that, in this case, $[1, w]^W$ is infinite. For Coxeter groups of type \tilde{A}_n , \tilde{C}_n , and \tilde{G}_2 , the interval $[1, w]^W$ was proved to be a lattice (in case \tilde{A}_n , this is true only for suitable choices of the Coxeter element) [Dig06, Dig12, McC15]. In the remaining irreducible cases, McCammond showed that the interval $[1, w]^W$ is not a lattice [McC15]. This led the way to a remarkable construction by McCammond and Sulway [MS17], which we summarize in the rest of this section.

Suppose from now on that W is an irreducible affine Coxeter group, acting as a reflection group on \mathbb{R}^n . Fix a Coxeter element w of W . Following [BM15], we say that an isometry of \mathbb{R}^n is *elliptic* if it fixes at least one point, and *hyperbolic* otherwise. The Coxeter element w is a hyperbolic isometry, and the points p of \mathbb{R}^n that minimize the Euclidean distance $d(p, w(p))$ form a line ℓ , called the *Coxeter axis* [McC15, Proposition 7.2]. There is a power of w which acts as a pure translation in the direction of ℓ . This direction is declared to be *vertical*, and the orthogonal directions are *horizontal*. An elliptic isometry is *horizontal* if it moves every point in a horizontal direction, and it is *vertical* otherwise [MS17, Definition 5.3]. Given $x \in [1, w]$, the *right complement* of x is the unique $y \in [1, w]$ such that $xy = w$; define the *left complement* similarly.

The coarse structure of the interval $[1, w]^W$ was described in [MS17, Definitions 5.4 and 5.5], using previous results on intervals in the group of Euclidean isometries [BM15]. The elements $u \in [1, w]^W$ are split into 3 rows according to the following cases (let v be the right complement of u):

- (bottom row) u is horizontal elliptic and v is hyperbolic;
- (middle row) both u and v are vertical elliptic;
- (top row) u is hyperbolic and v is horizontal elliptic.

The bottom and the top rows contain a finite number of elements, whereas the middle row contains infinitely many elements.

The reflection arrangement in \mathbb{R}^n associated with W splits \mathbb{R}^n into bounded simplicial chambers. Every point of the Coxeter axis is contained in the interior of some top-dimensional chamber, except for an infinite discrete set of equally spaced points. The chambers through which ℓ passes are called the *axial chambers*, and the vertices of these simplices are the *axial vertices*

[MS17, Definition 5.5]. The reflections which occur as edge labels in the interval $[1, w]^W$ are those that contain an axial vertex in their fixed hyperplane [McC15, Theorem 9.6]. This set includes the set R_V of all vertical reflections, but only a finite set R_H of horizontal reflections. Denote by T the (finite) set of translations in $[1, w]^W$ (they belong to the top row).

Since the interval $[1, w]^W$ is not a lattice in general, in [MS17, Section 6] new Euclidean isometries are introduced. They give rise to three more groups of isometries.

- The *diagonal group* D , generated by R_H and T . Translations are assigned a weight of 2.
- The *factorable group* F , generated by R_H and by a set T_F of *factored translations*. Factored translations are assigned a weight of $\frac{2}{k}$, where k is the number of irreducible components of the horizontal root system of W .
- The *crystallographic group* C , generated by R_H , R_V , and T_F .

The diagonal group D is included in both W and F , and all of them are included in the crystallographic group C . The associated intervals are related as follows [MS17, Lemma 7.2]:

$$\begin{aligned} [1, w]^C &= [1, w]^W \cup [1, w]^F \\ [1, w]^D &= [1, w]^W \cap [1, w]^F. \end{aligned}$$

The intervals $[1, w]^D$ and $[1, w]^F$ are finite, whereas $[1, w]^W$ and $[1, w]^C$ are infinite.

From these intervals, we can construct the interval groups D_w , F_w , and C_w . The group C_w is called the *braided crystallographic group*.

Theorem 9.3.4 ([MS17, Theorems 7.6 and 8.10]). The intervals $[1, w]^F$ and $[1, w]^C$ are lattices. Therefore, the interval groups F_w and C_w are Garside groups.

Theorem 9.3.5 ([MS17, Theorem 9.6]). The inclusions between the four intervals induce inclusions on the corresponding interval groups: $D_w \hookrightarrow W_w$, $D_w \hookrightarrow F_w$, $W_w \hookrightarrow C_w$, and $F_w \hookrightarrow C_w$. In addition, the braided crystallographic group C_w can be written as an amalgamated free product of W_w and F_w , amalgamated over D_w .

As a consequence, a finite-dimensional classifying space K_C for C_w can be constructed (e.g. via Definition 9.2.3). The cover of K_C corresponding to the subgroup W_w is a classifying space for the (dual) Artin group W_w [MS17, Proposition 11.1]. Therefore affine Artin groups admit a finite-dimensional classifying space. In [MS17, Section 11] it is asked to relate this space to the orbit configuration space Y_W (see Section 2.6), in order to establish the $K(\pi, 1)$ conjecture.

9.4 DUAL SALVETTI COMPLEX

Our aim is to compare the classifying spaces constructed in [MS17] and the orbit configuration space Y_W . Recall that Y_W is homotopy equivalent to the Salvetti complex X_W (Theorem 2.6.2). The first step is to define a complex $X'_W \simeq X_W$, with a structure which is closer to the dual Garside structures of [MS17]. For this reason, we call it the *dual Salvetti complex*.

Let W be an irreducible affine Coxeter group, and fix a Coxeter element $w = s_0 s_1 \cdots s_n$. The elements s_0, s_1, \dots, s_n are the reflections with respect to the walls of a fixed *base chamber*. The Coxeter axis passes through the base chamber, and so the base chamber is one of the (infinitely many) axial chambers. Denote by K_W the interval complex associated to $[1, w]^W$.

Definition 9.4.1 (Dual Salvetti complex). The Dual Salvetti complex X'_W is the (finite) subcomplex of K_W consisting of the simplices $[x_1 | x_2 | \cdots | x_d]$ such that $x_1 x_2 \cdots x_d$ is an elliptic isometry that fixes a vertex of the base chamber.

For every proper subset $Q \subsetneq S$, denote by w_Q the product of the elements of Q in the same relative order as in the list s_0, s_1, \dots, s_n . Then w_Q is a Coxeter element of the parabolic subgroup W_Q , and $w_Q \in [1, w]^W$ (the Hurwitz action [MS17, Definition 5.9] allows to construct a minimal length factorization of w that starts with w_Q). Denote by $\text{Fix}(u)$ the set of points fixed by the isometry u .

Lemma 9.4.2. We have $[1, w]^W \cap W_Q = [1, w_Q]^{W_Q}$, where the interval $[1, w_Q]^{W_Q}$ is constructed using all the reflections of W_Q as the generating set of W_Q .

Proof. The inclusion \supseteq follows directly from the fact that $w_Q \in [1, w]^W$. For the opposite inclusion, consider an element $u \in [1, w]^W \cap W_Q$. Since $u \in W_Q$, we have $\text{Fix}(w_Q) \subseteq \text{Fix}(u)$. By [BM15, Theorem 8.7], this implies that $u \leq w_Q$ in $[1, w]^W$. Then there is a minimum length factorization of w_Q (as a product of $|Q|$ reflections) that starts with a factorization of u . The reflections involved must belong to W_Q , and therefore $u \in [1, w_Q]^{W_Q}$. \square

For every $Q \subsetneq S$, the dual Salvetti complex has a subcomplex X'_{W_Q} consisting of the simplices $[x_1 | x_2 | \cdots | x_d]$ such that $x_1 x_2 \cdots x_d \in W_Q$. By Lemma 9.4.2, the subcomplex X'_{W_Q} is the interval complex associated to $[1, w_Q]^{W_Q}$. Then, by Theorems 9.3.3 and 9.2.2, it is a $K(G_{W_Q}, 1)$. Notice that, by definition, X'_W is the union of all subcomplexes X'_{W_Q} for $Q \subsetneq S$. Similarly, the Salvetti complex X_W is the union of the Salvetti complexes X_{W_Q} for $Q \subsetneq S$. Each X_{W_Q} is a $K(G_{W_Q}, 1)$, because the $K(\pi, 1)$ conjecture holds for spherical Artin groups [Del72].

Theorem 9.4.3. The dual Salvetti complex X'_W is homotopy equivalent to the Salvetti complex X_W .

Proof. Recall that the cells of the Salvetti complex X_W are indexed by the simplicial complex $\mathcal{K}_W = \{Q \subsetneq S\}$. Since $X_{W_Q} \simeq X'_{W_Q} \simeq K(G_{W_Q}, 1)$ for all $Q \subsetneq S$, it is possible to inductively construct homotopy equivalences $\varphi_Q: X_{W_Q} \rightarrow X'_{W_Q}$ satisfying the following naturality property: for all $P \subseteq Q \subset S$, there is a commutative diagram

$$\begin{array}{ccc} X_{W_P} & \xrightarrow{\varphi_P} & X'_{W_P} \\ \downarrow & & \downarrow \\ X_{W_Q} & \xrightarrow{\varphi_Q} & X'_{W_Q}. \end{array}$$

Gluing together all these maps, we obtain a map $\varphi: X_W \rightarrow X'_W$. This is a homotopy equivalence by a repeated application of the gluing theorem for adjunction spaces [Bro06, Theorem 7.5.7]. \square

Remark 9.4.4. The dual Salvetti complex can be defined for a general Artin group G_W , by gluing together the interval complexes of $[1, w_Q]^{W_Q}$ for Q in

$$\mathcal{K}_W = \{Q \subseteq S \mid \text{the parabolic subgroup } W_Q \text{ is finite}\}.$$

9.5 FINITE CLASSIFYING SPACES FOR BRAIDED CRYSTALLOGRAPHIC GROUPS

In this section we show how to construct a finite classifying space for the braided crystallographic group C_w .

Let K_C be the interval complex associated to $[1, w]^C$. This is a classifying space for C_w , by Theorems 9.3.4 and 9.2.2. Denote by $K_W \subseteq K_C$ the interval complex associated to $[1, w]^W$. Since the intervals $[1, w]^W$ and $[1, w]^C$ are infinite, the interval complexes K_W and K_C are also infinite.

Lemma 9.5.1. Let $\sigma = [x_1|x_2|\cdots|x_d] \in K_C$ be a d -simplex, with $d \geq 1$. Then exactly one of the following occurs:

- (i) every x_i is elliptic, and at least one is vertical;
- (ii) every x_i is horizontal elliptic or hyperbolic.

Proof. Let $y \in [1, w]^C$ be the right complement of $x_1x_2 \cdots x_d$.

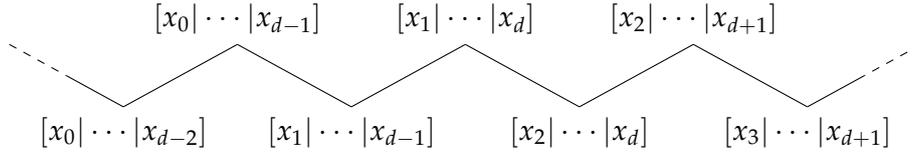
- If at least one x_i is not in $[1, w]^W$, then by [MS17, Lemma 7.2] no factorization of any x_i includes vertical reflections. In particular, no x_i is vertical elliptic, so (ii) holds and (i) does not. In the remaining cases, assume that $x_i \in [1, w]^W$ for all i .
- If every x_i is horizontal elliptic, then (ii) holds and (i) does not.

- Suppose that x_j is hyperbolic for some index j . Then $x_1 \cdots x_j$ is also hyperbolic, because a divisor of an elliptic isometry is elliptic. Therefore, by the coarse structure of $[1, w]^W$, the right complement $x_{j+1} \cdots x_d y$ is horizontal elliptic. Thus every x_i for $i > j$ is horizontal elliptic. Similarly, every x_i for $i < j$ is horizontal elliptic. Then (ii) holds and (i) does not.
- If there is at least one vertical elliptic element and there are no hyperbolic elements, (i) holds and (ii) does not. \square

Consider the poset map $\eta: K_C \rightarrow \mathbb{N}$ defined by

$$\eta([x_1|x_2|\cdots|x_d]) = \begin{cases} d & \text{if } x_1 x_2 \cdots x_d = w \\ d + 1 & \text{otherwise.} \end{cases}$$

We want to describe the connected components of a fiber $\eta^{-1}(d)$, viewed in the Hasse diagram of K_C . Let σ, τ be two simplices in the same fiber $\eta^{-1}(d)$. We have that τ is a face of σ if and only if $\sigma = [x_1|x_2|\cdots|x_d]$ with $x_1 x_2 \cdots x_d = w$ and either $\tau = [x_2|x_3|\cdots|x_d]$ or $\tau = [x_1|x_2|\cdots|x_{d-1}]$. Therefore, a connected component of $\eta^{-1}(d)$ has the following form:



where $x_i x_{i+1} \cdots x_{i+d-1} = w$ for all i . From now on, by d -fiber component (or simply fiber component) we mean a connected component of the fiber $\eta^{-1}(d)$. As described above, a d -fiber component has an associated sequence $(x_i)_{i \in \mathbb{Z}}$ of elements of $[1, w]^C$ such that the product of any d consecutive elements is w . This sequence is well-defined up to a translation of the indices.

Let $\varphi: [1, w]^C \rightarrow [1, w]^C$ be the conjugation by the Coxeter element w : $\varphi(u) = w^{-1} u w$. Notice that, in the sequence $(x_i)_{i \in \mathbb{Z}}$ associated to a d -fiber component, we have $\varphi(x_i) = x_{i+d}$ for all $i \in \mathbb{Z}$.

Lemma 9.5.2. Let $u \in [1, w]^C$. The set $\{\varphi^j(u) \mid j \in \mathbb{Z}\}$ is infinite if and only if u is vertical elliptic.

Proof. Recall from [MS17, Remark 7.3] the coarse structure of the interval $[1, w]^C$: an element $u \in [1, w]^C$ can be of type *bottom*, *middle*, *top*, or *factored*. The elements $\varphi^j(u)$ are all of the same type as u , and there is only a finite number of bottom, top and factored elements. Therefore, if $\{\varphi^j(u) \mid j \in \mathbb{Z}\}$ is infinite, u belongs to the middle row and therefore is vertical elliptic.

Let u be a vertical elliptic element, and suppose by contradiction that $\varphi^j(u) = u$ for some $j \in \mathbb{Z} \setminus \{0\}$. Let w^p be the smallest power of w which acts as a pure translation in the direction of the Coxeter axis. Then also $\varphi^{pj}(u) = u$,

so u commutes with the (non-trivial) translation w^{pj} . By [MS17, Lemma 11.3], $\text{Fix}(u)$ is invariant under w^{pj} . If $u = r_1 \cdots r_h$ is a minimal length factorization of u as a product of reflections, then $\text{Fix}(u) = \text{Fix}(r_1) \cap \cdots \cap \text{Fix}(r_h)$ [BM15, Lemma 6.4]. Thus every hyperplane $\text{Fix}(r_i)$ must be invariant under the translation w^{pj} . Then every reflection r_i is horizontal, so u is horizontal, and this is a contradiction. \square

Lemma 9.5.3. Let $\sigma = [x_1|x_2|\cdots|x_d] \in \eta^{-1}(d)$. The d -fiber component containing σ is finite if and only if σ is of type (ii) in Lemma 9.5.1.

Proof. If σ is of type (i), at least one x_i is vertical elliptic. Then the set $\{x_{i+jd} = \varphi^j(x_i) \mid j \in \mathbb{Z}\}$ is infinite by Lemma 9.5.2, so the component is infinite.

If σ is of type (ii), then every x_i is horizontal elliptic or hyperbolic. In either case, for all $i = 1, \dots, d$, the set $\{x_{i+jd} = \varphi^j(x_i) \mid j \in \mathbb{Z}\}$ is finite by Lemma 9.5.2. In other words, for all $i = 1, \dots, d$ the sequence $(x_{i+jd})_{j \in \mathbb{Z}}$ is periodic. Then the entire sequence $(x_i)_{i \in \mathbb{Z}}$ is periodic, and the component is finite. \square

Lemma 9.5.3 yields a classification of the finite d -fiber components. In this section we are interested in collapsing the infinite components. In particular, we want to prove that there is only a finite number of them.

Lemma 9.5.4. Let \mathcal{C} be an infinite d -fiber component. Then there exists a simplex $[x_1|x_2|\cdots|x_{d-1}] \in \mathcal{C}$ such that $x_1x_2 \cdots x_{d-1}$ is vertical elliptic.

Proof. Consider any d -simplex $[x_1|x_2|\cdots|x_d] \in \mathcal{C}$, with $x_1x_2 \cdots x_d = w$. Since \mathcal{C} is infinite, at least one x_i is vertical elliptic by Lemma 9.5.3. Suppose without loss of generality that x_d is vertical elliptic. Then its left complement $x_1x_2 \cdots x_{d-1}$ is also vertical elliptic. This completes the proof, because $[x_1|x_2|\cdots|x_{d-1}] \in \mathcal{C}$. \square

From now on, we are going to use some results of [McC15] which require W not to be of type \tilde{A}_n .

Lemma 9.5.5. Suppose that W is not of type \tilde{A}_n , and let $u \in [1, w]^W$ be an elliptic element. Then there is an axial vertex p such that $u \in [1, w_p]^{W_p}$, where W_p is the finite Coxeter subgroup of W that stabilizes p , and w_p is a Coxeter element of W_p . In particular, $\text{Fix}(u)$ contains at least one axial vertex p .

Proof. This proof is similar to the proof of [McC15, Proposition 9.5]. Let v be the right complement of u , so that $uv = w$. Let $v = r_1 \cdots r_m$ be a minimal factorization of v . Since u is elliptic, v is vertical and therefore at least one r_i is a vertical reflection. Using the Hurwitz action [McC15, Lemma 3.7], we can move this reflection to the end and thus assume that r_m is vertical. Then, by [McC15, Lemma 9.3], $wr_m = ur_1 \cdots r_{m-1}$ is a vertical elliptic isometry which is a Coxeter element for the finite Coxeter subgroup of W that stabilizes an axial vertex y . Then we conclude because $\text{Fix}(ur_1 \cdots r_{m-1}) \subseteq \text{Fix}(u)$ [BM15, Lemma 6.4]. \square

Lemma 9.5.6. Suppose that W is not of type \tilde{A}_n , and let \mathcal{C} be an infinite d -fiber component. There exists a simplex $[x_1|x_2|\cdots|x_{d-1}] \in \mathcal{C}$ such that $x_1x_2\cdots x_{d-1}$ is vertical elliptic and $\text{Fix}(x_1x_2\cdots x_{d-1})$ contains a vertex of the base chamber.

Proof. By Lemmas 9.5.4 and 9.5.5, there exists a simplex $[x_1|x_2|\cdots|x_{d-1}] \in \mathcal{C}$ such that $x_1x_2\cdots x_{d-1}$ is vertical elliptic and fixes an axial vertex. Recall from [McC15, Section 8] that every axial vertex can be written uniquely as $w^j(p)$ for some vertex p of the base chamber. Then, up to a conjugation by a power of w (i.e. up to a translation of the indices in the sequence $(x_i)_{i \in \mathbb{Z}}$), we can assume that $x_1x_2\cdots x_{d-1}$ fixes a vertex of the base chamber. \square

Remark 9.5.7. If $\text{Fix}(x_1x_2\cdots x_{d-1})$ contains a vertex of the base chamber, then $x_1x_2\cdots x_{d-1}$ is contained in a finite Coxeter group $W_Q \subseteq W$ generated by some $Q \subsetneq S$. Since $\text{Fix}(x_1x_2\cdots x_{d-1}) \subseteq \text{Fix}(x_i)$ by [BM15, Lemma 6.4], we also have $x_i \in W_Q$ for all $i = 1, \dots, d-1$.

Corollary 9.5.8. If W is not of type \tilde{A}_n , then there is only a finite number of infinite fiber components.

Proof. Every infinite d -fiber component contains a simplex $[x_1|x_2|\cdots|x_{d-1}]$ such that $x_1x_2\cdots x_{d-1}$ is vertical elliptic and $x_i \in W_Q$ for some (common) subset $Q \subsetneq S$, by Lemma 9.5.6 and Remark 9.5.7. There is only a finite number of such simplices because W_Q is finite. Therefore there is only a finite number of infinite d -fiber components. \square

Our aim is to show that K_C and K_W collapse onto finite subcomplexes $K'_C \subseteq K_C$ and $K'_W \subseteq K_W$, respectively. Notice that a fiber component $\mathcal{C} \subseteq K_C$ is either disjoint from K_W or entirely contained in K_W .

Definition 9.5.9. Let K be either K_C or K_W . A *nice subcomplex* of K is a subcomplex $K' \subseteq K$ such that

1. every finite fiber component $\mathcal{C} \subseteq K$ is also contained in K' ;
2. for every infinite fiber component $\mathcal{C} \subseteq K$, the intersection $\mathcal{C} \cap K'$ is non-empty and its Hasse diagram is connected.

Lemma 9.5.10. Let K be either K_C or K_W . The union of all finite fiber components of K forms a subcomplex of K .

Proof. By Lemma 9.5.3, it is equivalent to show that the simplices $\sigma \in K$ of type (ii) (see Lemma 9.5.1), together with the 0-simplex $[\]$, form a subcomplex of K . Let $[x_1|x_2|\cdots|x_d] \in K$ be a simplex of type (ii). Then every x_i is horizontal elliptic or hyperbolic. We have to check that all the simplices in the boundary of $[x_1|x_2|\cdots|x_d]$ are of type (ii) or equal to $[\]$. We only have to prove that no x_ix_{i+1} is vertical elliptic (for $i = 1, \dots, d-1$). If x_ix_{i+1} is vertical elliptic, then at least one of x_i and x_{i+1} is vertical and therefore hyperbolic. But a factorization of an elliptic element cannot include hyperbolic elements. \square

Theorem 9.5.11. Let K be either K_C or K_W .

- (a) K collapses onto every nice subcomplex K' .
- (b) If W is not of type \tilde{A}_n , finite nice subcomplexes of K exist.

Proof. For part (a), on every infinite fiber component \mathcal{C} consider the only acyclic matching $\mathcal{M}_{\mathcal{C}}$ with critical simplices given by $\mathcal{C} \cap K'$. Existence and uniqueness of $\mathcal{M}_{\mathcal{C}}$ follow from the fact that $\mathcal{C} \cap K'$ is non-empty and connected. The union of the matchings $\mathcal{M}_{\mathcal{C}}$ is a proper acyclic matching with the desired set of critical simplices.

For part (b), notice that there is only a finite number of (finite or infinite) fiber components. Then it is enough to inductively choose a finite non-empty interval in the Hasse diagram of every infinite d -fiber component \mathcal{C} , starting from $d = n + 1$ (the highest possible value of d) and going down to $d = 1$, so that every simplex in the boundary of a chosen simplex is also chosen. \square

Since K_C is a classifying space for C_w , we immediately obtain the following.

Theorem 9.5.12. If W is not of type \tilde{A}_n , the braided crystallographic group C_w admits a finite classifying space. \square

Recall from Definition 9.4.1 that the dual Salvetti complex X'_W is the subcomplex of K_W consisting of the simplices $[x_1|x_2|\cdots|x_d]$ such that $x_1x_2\cdots x_d$ fixes a vertex of the base chamber. We end this section by noticing that there is a canonical choice of a nice subcomplex K' of K (where K is either K_W or K_C): for every infinite fiber component, K' contains all the simplices between the first and the last simplex belonging to X'_W ; in addition, K' contains all the finite fiber components of K .

Lemma 9.5.13. Let K be either K_C or K_W , and assume that W is not of type \tilde{A}_n . Then K' is a nice subcomplex of K .

Proof. First we check that K' is a subcomplex. Let σ be a simplex of K' . If σ belongs to a finite fiber component, then all the faces of σ belong to finite fiber components, and therefore also belong to K' (see Lemma 9.5.10). Suppose instead that σ belongs to an infinite fiber component \mathcal{C} . Then, in \mathcal{C} , the simplex σ is between two simplices $\sigma_1, \sigma_2 \in X'_W$. It is easy to check that a face τ of σ is between two faces of σ_1 and σ_2 , in the fiber component of τ .

Condition 1 of Definition 9.5.9 is satisfied by construction. By Lemma 9.5.6, every infinite fiber component contains at least one simplex of X'_W , so condition 2 is also satisfied. \square

Remark 9.5.14. This canonical nice subcomplex contains the dual Salvetti complex X' as a subcomplex.

9.6 HORIZONTAL AND DIAGONAL GROUPS

In addition to the four groups of isometries defined in Section 9.3, the *horizontal group* H was also introduced in [MS17, Definition 6.8]. It is the subgroup of W generated by R_H . Notice that H is also a subgroup of the diagonal group D .

Recall that D_w is the interval group associated with the interval $[1, w]^D$. Following [MS17, Definition 7.5], let H_w be the group generated by R_H and subject only to the relations visible in $[1, w]^W$. This is not an interval group, because $w \notin H$. The group H_w is a subgroup of D_w by [MS17, Lemma 9.3].

The horizontal root system associated to W decomposes as $\Phi_{A_{n_1}} \cup \Phi_{A_{n_2}} \cup \dots \cup \Phi_{A_{n_k}}$, as in [MS17, Table 1]. The number k of horizontal components varies from 1 to 3. In the case $k = 1$, the factorable group F coincides with the diagonal group D , and the crystallographic group C coincides with the Coxeter group W . This happens in the cases \tilde{A}_n (but only for a suitable choice of the Coxeter element w), \tilde{C}_n , and \tilde{G}_2 .

Recall that the *noncrossing partitions of type B_m* are the noncrossing partitions of a regular $2m$ -gon that are symmetric with respect to the center. Also, the *zero block* of a noncrossing partition is the block containing the center of the $2m$ -gon. See [Arm09] for a general reference on this subject.

The following result explicitly describes a factorization of the groups H and H_w , and relates this factorization to the interval $[1, w]^F$.

Proposition 9.6.1 ([MS17, Proposition 7.6]). The interval $[1, w]^F$ is a direct product of k noncrossing partition lattices of types $B_{n_1+1}, \dots, B_{n_k+1}$. In addition, the groups H and H_w are explicitly determined as follows.

- H is a direct product of k Coxeter groups of types $\tilde{A}_{n_1}, \dots, \tilde{A}_{n_k}$;
- H_w is a direct product of k Artin groups of types $\tilde{A}_{n_1}, \dots, \tilde{A}_{n_k}$.

A consequence of Proposition 9.6.1 is that the horizontal part $H \cap [1, w]^W = H \cap [1, w]^F$ of the interval $[1, w]^W$ also splits as a product

$$H \cap [1, w]^W = \left(H_1 \cap [1, w]^W \right) \times \dots \times \left(H_k \cap [1, w]^W \right), \quad (9.1)$$

where H_1, \dots, H_k are the irreducible components of H . The factors are explicitly determined as follows.

Proposition 9.6.2. The poset $H_i \cap [1, w]^W$ is isomorphic to the subposet of the noncrossing partition lattice of type B_{n_i+1} consisting of the partitions without a zero block.

Proof. Let $m = n_i + 1$. As discussed in the proof of [MS17, Proposition 4.7], the boundary edges of the $2m$ -gon form a generating set for H_i . These elements generate exactly the noncrossing partitions without a zero block. \square

We now show how the group D_w is related to H_w . Together with Proposition 9.6.1, this allows to determine D_w explicitly.

Proposition 9.6.3. We have $D_w \cong \mathbb{Z} \rtimes H_w$, where \mathbb{Z} is the cyclic subgroup of D_w generated by w .

Proof. By [MS17, Remark 7.3], one possible generating set for D_w is given by $R_H \cup \{w\}$. The relations in D_w imply that $w^{-1}rw \in R_H$ for every $r \in R_H$. Therefore every element $u \in D_w$ can be written in the form $w^s h$ for some $s \in \mathbb{Z}$ and $h \in H_w$. Also, H_w is a normal subgroup of D_w .

As explained in the proof of [MS17, Lemma 9.4], the exponent s is uniquely determined by the image of u under the composition $D_w \rightarrow D \rightarrow \mathbb{Z}$. Here the first map is the natural projection, and the second map is the vertical displacement map, which sends all the horizontal reflections to 0 and the Coxeter element w to a non-zero integer. Therefore the decomposition $u = w^s h$ is unique. \square

9.7 FINITE CLASSIFYING SPACES FOR AFFINE ARTIN GROUPS

Let W be an irreducible affine Coxeter group. In this section we prove that the interval complexes K_D and K_W are classifying spaces. This is somewhat surprising, since the associated intervals $[1, w]^D$ and $[1, w]^W$ are not lattices in general. Then, using Theorem 9.5.11, we deduce that every affine Artin group admits a finite classifying space.

Let K_H be the subcomplex of K_D consisting of the simplices $[x_1|x_2|\cdots|x_d]$ such that the product $x_1 x_2 \cdots x_d$ is horizontal elliptic. Notice that the fundamental group of K_H is naturally isomorphic to H_w . Let $H \cong H_1 \times \cdots \times H_k$ be the factorization introduced in Section 9.6. Denote by K_{H_i} the subcomplex of K_H consisting of the simplices $[x_1|x_2|\cdots|x_d]$ such that the product $x_1 x_2 \cdots x_d$ belongs to the irreducible component H_i .

Lemma 9.7.1. K_H is homeomorphic to $K_{H_1} \times \cdots \times K_{H_k}$.

Proof. We claim that K_H is a triangulation of the natural cellular structure of $K_{H_1} \times \cdots \times K_{H_k}$. To show this, we explicitly construct a homeomorphism $\psi: K_{H_1} \times \cdots \times K_{H_k} \rightarrow K_H$. Consider a cell of $K_{H_1} \times \cdots \times K_{H_k}$, which is a product of simplices

$$[x_{11}|x_{12}|\cdots|x_{1d_1}] \times \cdots \times [x_{k1}|x_{k2}|\cdots|x_{kd_k}].$$

This is realized as $\Delta^{d_1} \times \cdots \times \Delta^{d_k} \subseteq \mathbb{R}^{d_1+\cdots+d_k}$. Consider a point $p \in \Delta^{d_1} \times \cdots \times \Delta^{d_k}$, with coordinates given by

$$(a_{11}, \dots, a_{1d_1}, \dots, a_{k1}, \dots, a_{kd_k}) \in \mathbb{R}^{d_1+\cdots+d_k}.$$

Assume for now that the coordinates of p are all distinct. Then there is a unique enumeration $\gamma: \{1, \dots, d_1 + \cdots + d_k\} \rightarrow \{11, \dots, 1d_1, \dots, k1, \dots, kd_k\}$

of the indices such that $a_{\gamma(1)} \geq a_{\gamma(2)} \geq \cdots \geq a_{\gamma(d_1+\cdots+d_k)}$. Notice that, in this enumeration, the relative order of indices in the same horizontal component is preserved. Define $\psi(p)$ as the point of $[x_{\gamma(1)}|x_{\gamma(2)}|\cdots|x_{\gamma(d_1+\cdots+d_k)}]$ with coordinates

$$(a_{\gamma(1)}, a_{\gamma(2)}, \dots, a_{\gamma(d_1+\cdots+d_k)}) \in \Delta^{d_1+\cdots+d_k}.$$

If not all the coordinates of p are distinct, then there are multiple choices for the enumeration γ , and any choice gives the same definition of $\psi(p)$. It is easy to check that ψ is well-defined and continuous. The fact that ψ is a homeomorphism follows from the decomposition (9.1). \square

As in Section 9.5, denote by $\varphi: [1, w]^W \rightarrow [1, w]^W$ the conjugation by w : $\varphi(u) = w^{-1}uw$.

Lemma 9.7.2. K_D is homeomorphic to $K_H \times [0, 1] / \sim$, where the relation \sim identifies $[x_1|x_2|\cdots|x_d] \times \{1\}$ and $[\varphi(x_1)|\varphi(x_2)|\cdots|\varphi(x_d)] \times \{0\}$ for all simplices $[x_1|x_2|\cdots|x_d] \in K_H$.

Proof. Let $Z = K_H \times [0, 1] / \sim$. Similarly to the proof of Lemma 9.7.1, we show that K_D is a triangulation of the natural cell structure of Z , by explicitly constructing a homeomorphism $\psi: Z \rightarrow K_D$. Notice that $K_H \times \{0\} \subseteq Z$ is naturally included into K_D , so we define $\psi|_{K_H \times \{0\}}$ as the natural homeomorphism with $K_H \subseteq K_D$. Consider now a cell of $K_H \times [0, 1]$ of the form

$$[x_1|x_2|\cdots|x_d] \times [0, 1].$$

This is realized as $\Delta^d \times [0, 1] \subseteq \mathbb{R}^{d+1}$. Then a point p in this cell has coordinates $(a_1, a_2, \dots, a_d, t)$, with $1 \geq a_1 \geq \cdots \geq a_d \geq 0$ and $t \in [0, 1]$. Let y be the right complement of $x_1x_2 \cdots x_d$, so that $x_1x_2 \cdots x_dy = w$. Notice that y is hyperbolic, so in particular $y \neq 1$. Assume for now that none of the coordinates a_1, \dots, a_d is equal to $1 - t$. Then there is a unique index $i \in \{0, \dots, d\}$ such that $a_1 \geq \cdots \geq a_i \geq 1 - t \geq a_{i+1} \geq \cdots \geq a_d$. Let $\sigma = [x_{i+1}|\cdots|x_d|y|\varphi(x_1)|\cdots|\varphi(x_i)]$. Define $\psi(p)$ as the point of σ with coordinates $(t + a_{i+1}, \dots, t + a_d, t, t + a_1 - 1, \dots, t + a_i - 1) \in \Delta^{d+1}$. If some of the coordinates a_1, \dots, a_d are equal to $1 - t$, choose any index i as above. Different choices of i give the same point of K_D .

It is easy to check that the definition of ψ on different cells is coherent, and that ψ is a homeomorphism. We only explicitly check that the definition on $[x_1|x_2|\cdots|x_d] \times [0, 1]$ agrees with the definition on $[x_1|x_2|\cdots|x_d] \times \{1\}$, since this is where the non-trivial gluing occurs. Consider a point $p \in [x_1|x_2|\cdots|x_d] \times \{1\}$, with coordinates $(a_1, \dots, a_d, 1) \in \Delta^d \times [0, 1]$.

- Since $[x_1|\cdots|x_d] \times \{1\}$ is identified with $[\varphi(x_1)|\cdots|\varphi(x_d)] \times \{0\}$, we have that $\psi(p)$ is the point of $[\varphi(x_1)|\cdots|\varphi(x_d)]$ with coordinates (a_1, \dots, a_d) .
- As an element of $[x_1|\cdots|x_d] \times [0, 1]$, the same point p is sent to the point of $[y|\varphi(x_1)|\cdots|\varphi(x_d)]$ with coordinates $(1, a_1, \dots, a_d)$. By definition of

the faces in an interval complex (Definition 9.2.3), this point is the same as the point of $[\varphi(x_1)| \cdots |\varphi(x_d)]$ with coordinates (a_1, \dots, a_d) .

Therefore the two definitions of ψ agree in this case. \square

Notice that Lemma 9.7.2 immediately implies Proposition 9.6.3. However, we found it more natural to prove Proposition 9.6.3 first.

Lemma 9.7.3. K_{H_i} is a classifying space for the affine Artin group of type \tilde{A}_{n_i} .

Proof. The isomorphism types of $H_i \cap [1, w]^W$ and K_{H_i} only depend on the rank n_i of the irreducible horizontal component H_i (see Proposition 9.6.2). Then we can assume, until the end of this proof, that W is of type \tilde{C}_n with $n = n_i + 1$. In this case $H = H_i$ is the unique irreducible horizontal component (see [MS17, Table 1]). Therefore $[1, w]^D = [1, w]^F$ is a lattice, D_w is a Garside group, and K_D is a classifying space for D_w .

By Lemma 9.7.2, there is a covering map

$$\rho: K_H \times \mathbb{R} \rightarrow K_D$$

which corresponds to the subgroup K_H of K_D . Then $K_H \times \mathbb{R}$ is a classifying space. Since $K_H \times \mathbb{R} \simeq K_H = K_{H_i}$, we have that K_{H_i} is also a classifying space. \square

Remark 9.7.4. The complex K_{H_i} is closely related to the dual Salvetti complex of type \tilde{A}_{n_i} , introduced in Section 9.4. Indeed, K_{H_i} is obtained by gluing dual Salvetti complexes corresponding to the maximal parabolic subgroups. However, the Coxeter elements for these dual Salvetti subcomplexes are not divisors of a common Coxeter element of \tilde{A}_{n_i} .

Theorem 9.7.5. K_D is a classifying space for D_w .

Proof. By Lemma 9.7.1, we have that $K_H \cong K_{H_1} \times \cdots \times K_{H_k}$. Each factor is a classifying space by Lemma 9.7.3, therefore K_H is a classifying space for H_w . As discussed in the proof of Lemma 9.7.3, by Lemma 9.7.2 there is a covering map $\rho: K_H \times \mathbb{R} \rightarrow K_D$. Therefore K_D is a classifying space. \square

Theorem 9.7.6. K_W is a classifying space for the affine Artin group $G_W \cong W_w$.

Proof. Recall from Section 9.3 the four intervals $[1, w]^D, [1, w]^W, [1, w]^F, [1, w]^C$, to which we can associated the interval complexes K_D, K_W, K_F, K_C . Since $[1, w]^F$ and $[1, w]^C$ are lattices, the corresponding complexes K_F and K_C are classifying spaces. Consider the universal cover of K_C :

$$\rho: \tilde{K}_C \rightarrow K_C.$$

9.8. The $K(\pi, 1)$ conjecture for the Artin group of type \tilde{D}_4

By [MS17, Lemma 7.2], we have that $K_D = K_W \cap K_F$ and $K_C = K_W \cup K_F$. Therefore $\rho^{-1}(K_D) = \rho^{-1}(K_W) \cap \rho^{-1}(K_F)$ and $\tilde{K}_C = \rho^{-1}(K_C) = \rho^{-1}(K_W) \cup \rho^{-1}(K_F)$. Then there is a Mayer-Vietoris long exact sequence

$$\cdots \rightarrow H_i(\rho^{-1}(K_D)) \rightarrow H_i(\rho^{-1}(K_W)) \oplus H_i(\rho^{-1}(K_F)) \rightarrow H_i(\tilde{K}_C) \rightarrow \cdots$$

where all homology groups are with integer coefficients. Since $D_w = \pi_1(K_D)$ is a subgroup of $C_w = \pi_1(K_C)$, we have that $\rho^{-1}(K_D)$ is a union of (infinitely many) disjoint copies of the universal cover \tilde{K}_D of K_D . Similarly, $\rho^{-1}(K_W)$ is a union of disjoint copies of the universal cover \tilde{K}_W of K_W , and $\rho^{-1}(K_F)$ is a union of disjoint copies of the universal cover \tilde{K}_F of K_F . Since K_F and K_C are classifying spaces, both \tilde{K}_F and \tilde{K}_C are contractible. By Theorem 9.7.5, \tilde{K}_D is also contractible. Then, for $i \geq 1$, the homology groups $H_i(\rho^{-1}(K_D))$, $H_i(\rho^{-1}(K_F))$, and $H_i(\tilde{K}_C)$ vanish. By the Mayer-Vietoris long exact sequence, $H_i(\rho^{-1}(K_W))$ also vanishes for $i \geq 1$. This means that \tilde{K}_W has a trivial reduced homology, so it is contractible by [Hat02, Corollary 4.33]. \square

Theorem 9.7.7. Every affine Artin group admits a finite classifying space.

Proof. Let G_W be an affine Artin group. If it is of type \tilde{A}_n , then the $K(\pi, 1)$ conjecture holds [Oko79, CP03], and the Salvetti complex X_W is a finite classifying space for G_W . If G_W is not of type \tilde{A}_n , then by Theorem 9.5.11 the complex K_W can be collapsed onto a finite subcomplex K'_W . We know that K_W is a classifying space for G_W by Theorem 9.7.6, so the same is true for K'_W . \square

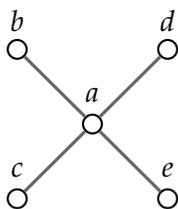
Remark 9.7.8. In order to prove the $K(\pi, 1)$ conjecture for an affine Artin group G_W , it is enough to show that K_W (or K'_W) collapses onto the dual Salvetti complex X'_W . In Section 9.8 we prove this in the case \tilde{D}_4 . We believe that this can be done in general.

9.8 THE $K(\pi, 1)$ CONJECTURE FOR THE ARTIN GROUP OF TYPE \tilde{D}_4

In this section we prove the $K(\pi, 1)$ conjecture for the Artin group of type \tilde{D}_4 . Notice that this is one of the two 4-dimensional affine cases for which the $K(\pi, 1)$ conjecture was still not proved, together with \tilde{F}_4 (see Section 2.6). We also describe how the most important constructions of this chapter specialize to the case \tilde{D}_4 .

Let W be the Coxeter group of type \tilde{D}_4 , with generators labeled as in Figure 9.2. It can be realized as a reflection group in \mathbb{R}^4 by associating the generators a, b, c, d, e with the reflections with respect to the hyperplanes

$$\begin{aligned} H_a &= \{x_1 + x_2 + x_3 + x_4 = 1\} \\ H_b &= \{x_1 = 0\} \\ H_c &= \{x_2 = 0\} \\ H_d &= \{x_3 = 0\} \end{aligned}$$


 Figure 9.2: Coxeter graph of type \tilde{D}_4 .

$$H_e = \{x_4 = 0\}.$$

The base chamber is the 4-simplex with vertices $(0, 0, 0, 0)$, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$. Notice that the last four vertices form a regular 3-dimensional tetrahedron.

Take $w = abcde$ as the Coxeter element. It acts on \mathbb{R}^4 as the composition of a translation by $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and an axial symmetry with respect to the line $\ell = \langle 1, 1, 1, 1 \rangle$. In particular, this line is the Coxeter axis. By definition, its direction is the vertical direction, and the directions perpendicular to it are horizontal.

The convex hull of the axial vertices is an infinite prism with a cubical base. More precisely, its intersection with the hyperplane H_a is a 3-dimensional cube \mathbf{C} with vertices

$$\begin{aligned} & (1, 0, 0, 0), \quad (0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1), \\ & (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}). \end{aligned}$$

The convex hull of the axial vertices is then obtained by translating this cube in the vertical direction.

The horizontal elliptic isometries of $[1, w]^W$ (which form the bottom row of the coarse structure) are easily described in terms of the cube \mathbf{C} :

- the identity;
- 6 horizontal reflections with respect to the 6 faces of \mathbf{C} (each fixes a hyperplane in \mathbb{R}^4);
- 12 horizontal symmetries with respect to the 12 edges of \mathbf{C} (each fixes a 2-dimensional subspace of \mathbb{R}^4);
- 8 horizontal symmetries with respect to the 8 vertices of \mathbf{C} (each fixes a 1-dimensional subspace of \mathbb{R}^4 , parallel to the Coxeter axis).

Denote by h_i^\pm the 6 horizontal reflections, for $i \in \{1, 2, 3\}$, so that h_i^+ and h_i^- are reflections with respect to parallel hyperplanes.

The horizontal root system has three irreducible components of type A_1 , which correspond to the three pairs of opposite faces of \mathbf{C} . Then the group

9.8. The $K(\pi, 1)$ conjecture for the Artin group of type \tilde{D}_4

H_w is a direct product of three copies of the Artin group of type \tilde{A}_1 (see Proposition 9.6.1):

$$H_w \cong (\mathbb{Z} * \mathbb{Z})^3.$$

The free generators of the i -th direct factor $\mathbb{Z} * \mathbb{Z}$ are h_i^+ and h_i^- . The complex K_H is a triangulation of $(S^1 \vee S^1)^3$, which is a classifying space for H_w (see Lemma 9.7.1).

The diagonal interval group D_w is a semidirect product $\mathbb{Z} \ltimes (\mathbb{Z} * \mathbb{Z})^3$, where the leftmost \mathbb{Z} factor is generated by w (see Proposition 9.6.3). The conjugation by w exchanges h_i^+ and h_i^- for $i \in \{1, 2, 3\}$. Then w acts diagonally on each factor $\mathbb{Z} * \mathbb{Z}$ by exchanging the two free generators h_i^\pm . By Lemma 9.7.2, the complex K_D is a triangulation of

$$(S^1 \vee S^1)^3 \times [0, 1] / \sim,$$

where the relation \sim glues $(S^1 \vee S^1)^3 \times \{0\}$ with $(S^1 \vee S^1)^3 \times \{1\}$ by exchanging the two S^1 's in each of the three factors.

The dual Salvetti complex X'_W can be explicitly constructed with a computer. It has dimension 4, and its f -vector is $(1, 134, 806, 1344, 672)$. Notice that it is much bigger than the usual Salvetti complex X_W , which has f -vector $(1, 5, 10, 10, 5)$. The canonical nice subcomplex $K'_W \subseteq K_W$, defined at the end of Section 9.5, can be constructed from X'_W by extending the fiber components. It has dimension 5, and its f -vector is $(1, 173, 1389, 3388, 3252, 1080)$. As shown in Section 9.7, K'_W is a classifying space for the Artin group G_W . In order to prove the $K(\pi, 1)$ conjecture, it is enough to check that K'_W collapses onto X'_W . In the case \tilde{D}_4 we checked it by computer, using the algorithm described in [BL14]. This proves the following theorem, which is dedicated to my wife Aleksandra.

Theorem 9.8.1. The $K(\pi, 1)$ conjecture holds for the Artin group of type \tilde{D}_4 .

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Acknowledgements

The first big *thank you* goes to my supervisor, Mario Salvetti, who has been wise, friendly, calm, and kind. We have shared many projects in the past few years. He is absolutely a great person to meet and to work with.

I would like to thank my other coauthors: Emanuele Delucchi, Noriane Girard, and Davide Lofano. They are all incredibly nice collaborators. Then there is a longer list of mathematicians who gave thoughts and suggestions on the topics of this thesis: Christin Bibby, Federico Glaudo, Alessandro Iraci, Daniele Semola, Simona Settepanella, Tan Nhat Tran, and Masahiko Yoshinaga. I truly appreciated their help. I am also glad I worked together with Lorenzo Guerra and Roberto Pagaria, and I thank them for it.

Finally, I am really grateful to Aleksandra Baranova and Ugo Bindini for their aesthetic and typographical advice, as well as for proofreading this thesis.



Desidero ora esprimere la mia riconoscenza a tutte le persone che hanno contribuito a rendere questi anni di dottorato tra i più felici della mia vita.

Il massimo ringraziamento va ad Aleksandra, per tutto ciò che ha fatto, che fa, e che farà per me. Non esistono parole in italiano, russo, elfico, o in alcuna altra lingua, per descrivere la gratitudine che merita. Ci sono tre cose che ho scoperto durante il dottorato e che desidero condividere con voi; la prima è questa: sposarsi è meraviglioso!

Vorrei ringraziare i miei genitori, che pensano sempre a me, per quanto distante io sia, e che hanno sempre sostenuto e incoraggiato le mie scelte. Sono fortunato ad essere loro figlio. Ringrazio anche i miei nonni e il resto della mia famiglia,¹ per essermi sempre vicini, anche da lontano.

Con gli amici di Pisa ho condiviso più esperienze di quante si possano raccontare. Comincio ringraziando, in rigoroso ordine sparso: Ugo, Margherita, Sasha, Venez, Tess, Alice, Davide, Federico, Giada, Davide, Alessandra, Francesco, Luca, Lucrezia, Julian, Cristoforo, Gigi. Grazie per la compagnia, il supporto, il divertimento, il tempo perso.² Grazie per le partite a Bridge,

¹Sia in Italia che in Russia!

²Perdere tempo nel modo giusto è una delle cose che rendono interessante la vita.

Subotto, Hanabi, Scacchi, Kubb, Pandemic,³ Charterstone, Avventure nella Terra di Mezzo, e tutto il resto. Grazie per le cene, *mensa e caffè*, le vacanze a Ravaschetto, i trailer, i playtest. La seconda cosa che ho scoperto in questi anni, e che vorrei condividere con voi, è questa: dei buoni amici sono più importanti di una casa grande.⁴

Continuo ringraziando alcuni altri amici, che perlopiù hanno smesso di abitare a Pisa ma che sono riuscito a rivedere almeno occasionalmente: Luca, Alessandro, Massimo, Gio, Irene, Michele, Sara, Fabrizio, Marco, Gennady, Giulia, Stefania, Vittorio. Grazie per i matrimoni, i ritrovi, e i progetti (in particolare, il Subtracker). A tal proposito, ecco la terza cosa che ho scoperto durante il dottorato: il Subotto aiuta a trovare lavoro.⁵

Non tutti gli amici vengono da Pisa! Vorrei ringraziare la mia vecchia compagnia di Brescia, che rivedo sempre molto volentieri: Lorenzo, Federico, Silvia, Davide, Gabriele, Francesco, Monica, e tutti gli altri. Grazie per i compleanni, le partite, le cene, e le gare di programmazione. Ringrazio inoltre gli amici di Friburgo: Linard, Alessio, Elia, Farhad, ed Emanuele. Spero di rivederli presto!

Molti altri amici, eventi, e storie meriterebbero menzione: l'ICPC, a cui ho partecipato grazie a Federico, Francesco, e Filippo;^{6,7} gli IDEAG; i Summer Math Camp;⁸ Caldé; il coro dell'Università di Pisa; la scuola di tango. Tutto ciò ha contribuito a rendere gli ultimi anni ancora più pieni.

Con questa tesi si conclude la mia esperienza alla Scuola Normale, a cui devo tantissimo. Ringrazio questa splendida istituzione, i suoi allievi, i suoi docenti, e tutti coloro che vi hanno lavorato durante la mia permanenza a Pisa. Quantunque otto anni trascorsi in mezzo a voi siano davvero troppo pochi, ora è giunta la fine. Addio!

³Ivi inclusi Pandemic Legacy Season 1, Pandemic Legacy Season 2, la malattia viola, il ceppo virulento, il bioterrorista, il laboratorio, e la partita con 10 Epidemic.

⁴Ma una casa grande aiuterebbe, specialmente quando i buoni amici sono numerosi!

⁵Grazie Stefano!

⁶Il segreto di un coach forte è una squadra forte.

⁷Forse a questo punto dovrei ringraziare anche Verdi e Chopin.

⁸Sono sempre contento di partecipare, in primis per l'amicizia con Enrico, Sandro, Salvatore, Carlo, e Luca.