How to collapse a simplicial complex

Theory and practice

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Fundamental questions about X

Is X trivial?

Compute invariants of X

Is X the same as Y?

And what is X, exactly?

"Forty-two," said Deep Thought, with infinite majesty and calm.

If X is a topological space...

Is X contractible?

What are the homology groups of X?

Are X and Y homotopy equivalent?

And what is the homotopy type of X, exactly?

Simplicial complexes

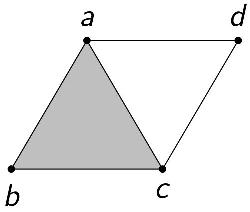
Let V be a (finite) set of *vertices*.

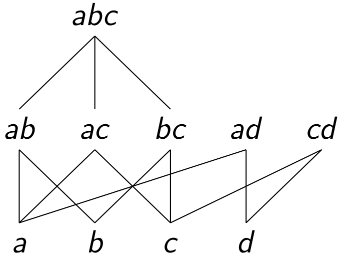
A *simplicial complex* is a collection $X \subseteq 2^V \setminus \{\emptyset\}$ such that

$$\sigma \in X \text{ and } \emptyset \neq \tau \subseteq \sigma \implies \tau \in X.$$

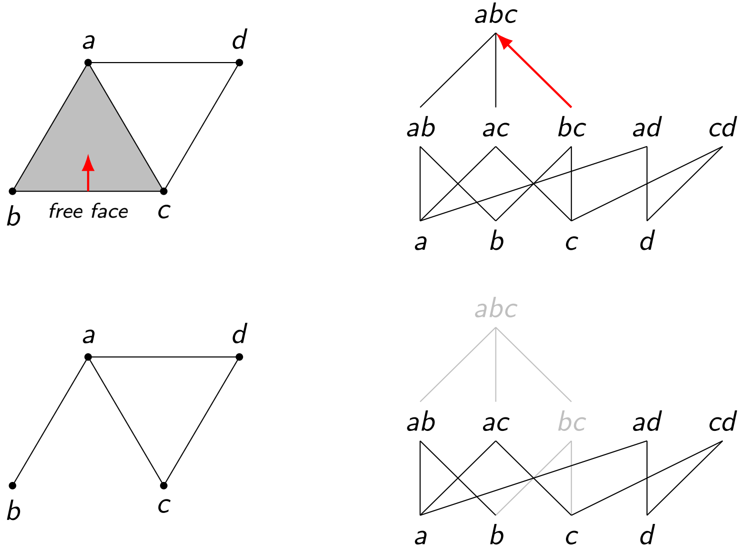
Example

$V = \{a, b, c, d\}$
$X = \{abc, ab, ac, ad, bc, cd, a, b, c, d\}$

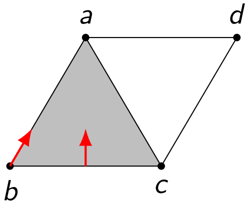


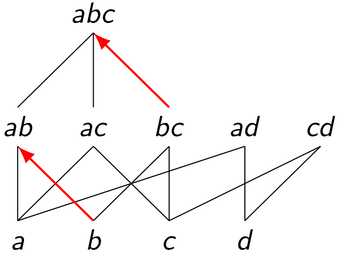


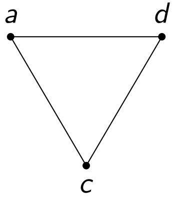
Elementary collapse

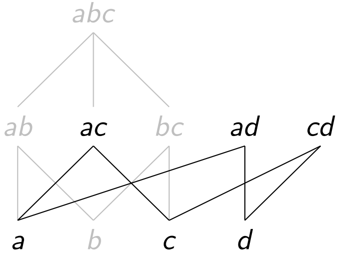


A sequence of elementary collapses

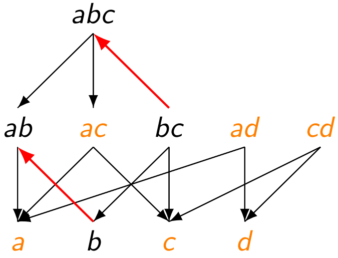








Discrete Morse theory

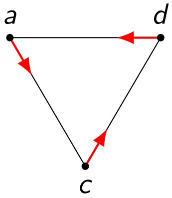


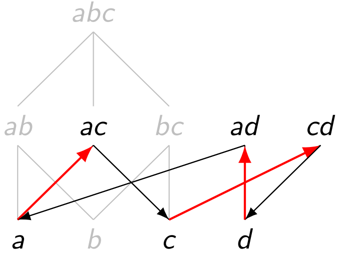
Main theorem of discrete Morse theory (Forman '98, Chari '00)

Let $\mathcal{M}$ be an *acyclic* matching on the face poset of X such that the *critical* simplices form a subcomplex X^*.

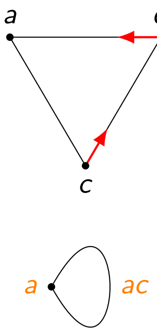
Then X deformation retracts onto X^* through a sequence of elementary collapses.

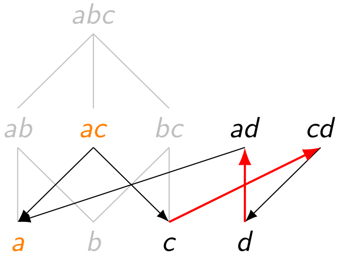
A non-acyclic matching





A triangle does not deformation retract onto an empty complex.





Main theorem of discrete Morse theory (second version)

Let $\mathcal{M}$ be an *acyclic* matching on the face poset of X. Then X is homotopy equivalent to X', a CW complex with cells in bijection with the critical simplices.

More versions of discrete Morse theory

- X can be a CW complex.
  Elementary collapses are only allowed for *regular* faces.

- X can be infinite (Batzies '02).
  An additional compactness condition is needed on the matching $\mathcal{M}$.

this should be avoided

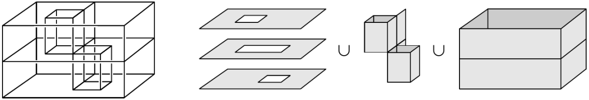
- X can be an algebraic chain complex.
  - Free (Jöllenbeck-Welker '05, Kozlov '05, Sköldberg '06)
  - Torsion (Salvetti-Villa '13, P.-Salvetti '18)

Collapsibility

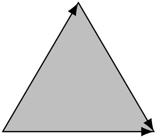
X is *collapsible* if it admits a sequence of elementary collapses leaving a single vertex.

Equivalently: its face poset admits an acyclic matching with only one critical simplex (a vertex).

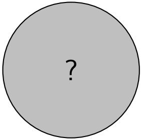
Collapsible implies contractible, but the converse is not true.



Bing's *house with two rooms* (image from Hatcher '01)



Zeeman's *dunce hat*



Non-collapsible 3-balls (Benedetti '12)

Back to our questions about X

Is X contractible?

Check if X is collapsible

What are the homology groups of X?

Use (algebraic) discrete Morse theory on the chain complex

Are X and Y homotopy equivalent?

If $Y \subseteq X$, check if X collapses onto Y

And what is the homotopy type of X, exactly?

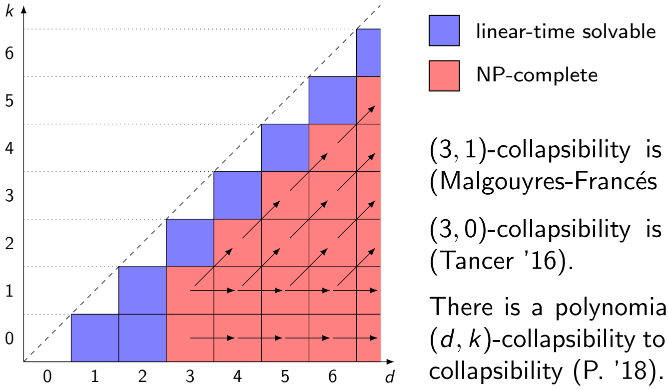
Find an optimal acyclic matching, and the Morse complex might be simple enough (e.g. a wedge of spheres)

Algorithmic questions (and answers)

- Contractibility of a 4-dimensional simplicial complex is undecidable (Novikov). Open problem in dimensions 2 and 3.
  - Construct manifolds M such that M is a ball if and only if $\pi_1(M)$ is trivial.
- Collapsibility of a 2-dimensional simplicial complex is solvable in linear time.
  - Collapse greedily. If you get stuck, the complex is not collapsible.
- Finding an optimal acyclic matching on a 2-dimensional simplicial complex is NP-hard (Eğecioğlu and Gonzalez '96).
  - Reduction from the vertex cover problem.
- Collapsibility of a 3-dimensional simplicial complex is NP-complete (Malgouyres-Francés '08, Tancer '16).
  - Reduction from 3-SAT. Gadgets are based on Bing's house.

(d, k)-collapsibility

(d, k)-collapsibility: Determine whether a d-dimensional simplicial complex collapses onto a k-dimensional subcomplex.



$(3, 1)$-collapsibility is NP-complete (Malgouyres-Francés '08).

$(3, 0)$-collapsibility is NP-complete (Tancer '16).

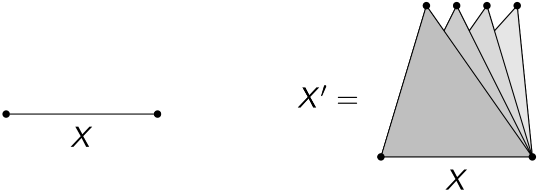
There is a polynomial reduction of (d, k)-collapsibility to $(d+1, k+1)$-collapsibility (P. '18).

Therefore, if $d \geq k+2$ and $d \geq 3$, (d, k)-collapsibility is NP-complete.

Reduction $(d, k) \to (d+1, k+1)$

Let X be an instance of (d, k)-collapsibility, i.e., a d-dimensional simplicial complex.

Construct X' by taking $n+1$ copies $C_1, \ldots, C_{n+1}$ of the cone over X, all glued together on the base X, where $n = |X|$.



- If X is (d, k)-collapsible, then collapse the cone C_1 onto its apex and all other cones $C_i \setminus X \cong X \cup \{\emptyset\}$ as X.
- If X' is $(d+1, k+1)$-collapsible, at least one $C_i \setminus X$ has no simplices matched with a simplex of X. Then collapse X as $C_i \setminus X$.

Thank you!

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